Safety-Constrained Learning and Control using Scarce Data and Reciprocal Barriers ⋆

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Abstract

We develop a control algorithm that ensures the safety, in terms of confinement in a set, of a system with unknown, 2nd-order nonlinear dynamics. The algorithm establishes novel connections between data-driven and robust, nonlinear control. It is based on data obtained online from the current trajectory and the concept of reciprocal barriers. More specifically, it first uses the obtained data to calculate set-valued functions that over-approximate the unknown dynamic terms. For the second step of the algorithm, we design a robust control scheme that uses these functions as well as reciprocal barriers to render the system forward invariant with respect to the safe set. In addition, we provide an extension of the algorithm that tackles issues of controllability loss incurred by the nullspace of the control-direction matrix. The algorithm removes a series of standard, limiting assumptions considered in the related literature since it does not require global boundedness, growth conditions, or a priori approximations of the unknown dynamics' terms.

Key words: Uncertain systems; Iterative learning control; Constrained control; Nonlinear systems; Robust control

1 Introduction

Learning-based control is an important emerging topic of research that tackles uncertain autonomous systems. Control of uncertain systems has been widely studied in the literature, mostly by means of robust and adaptive control [1]. These techniques, however, require restrictive assumptions on the uncertainty type, such as linear parametrizations, a priori neural-network approximations, or additive disturbances. Such assumptions might be too restrictive in cases where the dynamics sustain abrupt unknown changes, due to, for instance, unpredicted failures. Traditional control techniques might fail in such scenarios and one must turn to data-based approaches. At the same time, since we aim to tackle cases of abrupt dynamic changes, standard episodic reinforcement learning algorithms [2] are inapplicable; we are restricted to data obtained on the fly from the current trajectory, which limits greatly the available resources.

This paper considers the problem of safety, in the sense of confinement in a given set, of 2nd-order *nonlinear systems* of the form (to be precisely defined in Sec. 2)

$$\dot{x}_1 = x_2 \tag{1a}$$
$$\dot{x}_2 = f(x) + g(x)u \tag{1b}$$

with *a priori unknown* terms f and g, for which the assumptions we impose are restricted to local Lipschitz continuity. Unlike previous works in the related literature, we do not impose growth conditions [3] or *global* Lipschitz continuity on the dynamics [4, 5], and we do not assume boundedness of the state [5]. Moreover, we do not restrict g to be in the class of square positive definite matrices, a convenient property that has been commonly used in the related literature [6–8]. Finally, we do not employ approximations of the dynamics, such as linear parametrizations [9, 10] or neural networks [6].

Our proposed solution consists of a two-layered algorithm for the safety control of the unknown system in (1), integrating nonlinear feedback control with on the fly data-driven techniques. More specifically, the main contributions are as follows. Firstly, we use a discrete, finite set of data obtained from the current trajectory to compute an estimate of the control matrix g. Secondly, we use this estimate to design a novel feedback control protocol based on reciprocal barriers, rendering the system forward-invariant with respect to the given safe set under certain assumptions on the estimation error. We further provide an analytic relation between the estimation error and the frequency of the obtained data. Fi-

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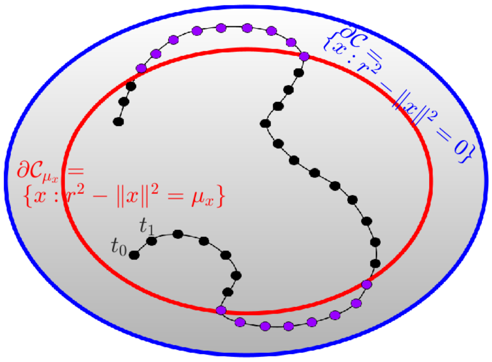

Fig. 1. Example of a set $\mathcal{C} = \{x \in \mathbb{R}^n : h(x) = r^2 - \|x\|^2 > 0\}$ for the invariance of the system trajectory; $\mathcal{C}_{\mu_x} = \{x \in \mathbb{R}^n : h(x) = r^2 - \|x\|^2 \in (0, \mu_x)\}$ dictates the region where the safety controller is activated.

nally, we provide a provably correct extension that tackles controllability loss incurred by the control matrix g. The proposed algorithm is "minimally invasive", in the sense that it acts only close to the boundary of the safe set, and does not require any expensive numerical operations or tedious analytic expressions to produce the control signal, enhancing thus its applicability.

This paper extends our preliminary work [11] in the following directions: firstly, we consider second-order systems allowing position-like safety constraints with respect to x_1, in contrast to the first-order case of [11]. Secondly, we provide better insight to the proposed solution by relating the estimation error of g with the frequency of the obtained data. Finally, we provide an analytic proof for the correctness of the algorithm that tackles the controllability loss incurred by the control matrix g.

Related Work: Safety of autonomous systems [12] has been studied intensively by the control community. The most widely used methodology is the concept of barrier certificates [13], which provide a convenient and efficient way to guarantee invariance in a given set [14–16]. Nevertheless, standard control based on barrier certificates relies heavily on the underlying dynamics since the respective terms are used in the control design. Extensions that tackle dynamic uncertainties have been considered in [9, 10, 17–19] using robust and adaptive control, restricted, however, to additive perturbations or constant unknown parameters. Similarly, [7, 20–22] use barrier functions to guarantee safety for systems whose dynamic terms satisfy linear parametrizations with respect to uncertainties or growth and dissipative conditions. Therefore, the respective methodologies are not applicable to the class of systems considered in this paper.

Another class of works dealing with unknown dynamics is that of funnel control, which guarantees confinement of the state in a given funnel, [6, 23, 24]. In contrast to the setup of the current paper, such methodologies either rely on approximation of the dynamics using neural networks [6], or require positive definite input matrix g [6, 24] (see (1)). The former lacks good heuristics for choosing radial basis functions and number of layers and relies on strong assumptions on the amount of data and the approximation errors, while the latter restricts the class of considered systems.

A promising family of methodologies that deals with unknown dynamics is that of data-driven control [25–29]. Most of the related works, however, provide theoretically verified results for the limiting case of linear dynamics and consider the problem of stabilization/tracking, without being able to account for safety constraints that concern the transient state of the system. Learning-based, data-driven approaches have also been integrated with barrier certificates to address the safety of uncertain systems [8, 30–35]; [30–32], however, only consider additive uncertain terms modeled by Gaussian processes and assumed to evolve in compact sets. In [35] the authors have access to a nominal model and propose an episodic reinforcement learning approach that tackles the residual disturbance. [34] and [33] use data for learning barrier functions by employing the underlying dynamics, either partially or fully, and [36] considers the safety problem for stochastic system with additive disturbances. Finally, [8] uses approximation of the dynamics using neural networks. On the contrary, we consider nonlinear systems of the form (1) where both f and g are entirely unknown.

Moreover, many of the aforementioned works require large amounts of data in order to provide accurate results. Recent methodologies that employ limited data obtained on the fly have been developed in [4, 5, 37], imposing, however, restrictive assumptions on the dynamics, such as global boundedness and Lipschitz continuity with known bounds, or known bounds on the approximation errors. In addition, the aforementioned works resort to online optimization techniques for safety specifications, increasing thus the complexity of the resulting algorithms. Other standard optimization-based algorithms that guarantee safety through state constraints [26, 38] cannot tackle dynamic uncertainties more sophisticated than additive bounded disturbances. In the current paper, we rely on limited data without imposing any of the assumptions stated above, developing a computationally efficient safety control algorithm.

The remainder of this article is structured as follows. Section 2 gives the problem formulation. Section 3 presents the approximation algorithm and the control design is provided in Section 4. Section 5 investigates the case of controllability loss, and Section 6 presents simulation examples. Finally, Section 7 concludes the paper. The proofs of our results are provided in the Appendix.

2 Problem Formulation

Notation: We denote by $\mathbb{N}$ and $\bar{\mathbb{N}} \coloneqq \mathbb{N} \cup \{0\}$ the sets of naturals and nonnegative integers, respectively. The set of n-dimensional nonnegative reals, with $n \in \mathbb{N}$, is denoted by $\mathbb{R}^n_{\geq 0}$; $\text{Int}(A)$, ∂A, and $\text{Cl}(A)$ denote the interior, boundary, and closure, respectively, of a set $A \subseteq \mathbb{R}^n$. The open and closed ball of radius $r > 0$ around $x \in \mathbb{R}^n$ is denoted by $\mathcal{B}_r(x)$ and $\bar{\mathcal{B}}_r(x)$, respectively. The minimum eigenvalue of a matrix $A \in \mathbb{R}^{n\times m}$ is denoted by $\lambda_{\min}(A)$. Given $a \in \mathbb{R}^n$, $\|a\|$ denotes its 2-norm; $\nabla_y h(\cdot) \coloneqq \frac{\partial h(\cdot)}{\partial y} \in \mathbb{R}^m$ is the gradient of a real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ with respect to $y \in \mathbb{R}^m$, and $\nabla h(x) \coloneqq \frac{\mathrm{d}h(x)}{\mathrm{d}x} \in \mathbb{R}^n$. An interval in $\mathbb{R}$ is denoted by $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and the set of intervals on $\mathbb{R}$ by $\mathbb{IR} \coloneqq \{\mathcal{A} \coloneqq [\underline{\mathcal{A}}, \bar{\mathcal{A}}] : \underline{\mathcal{A}}, \bar{\mathcal{A}} \in \mathbb{R}, \underline{\mathcal{A}} \leq \bar{\mathcal{A}}\}$, which extends to the sets of interval vectors $\mathbb{IR}^n$ and matrices $\mathbb{IR}^{n\times m}$. We denote by $|\mathcal{A}| \coloneqq \max\{\overline{\mathcal{A}}, \underline{\mathcal{A}}\}$ the absolute value of an interval $\mathcal{A} \in \mathbb{IR}$, and the infinity norm of $\mathcal{B} \coloneqq (\mathcal{B}_1, \dots, \mathcal{B}_n) \in \mathbb{IR}^n$ by $\|\mathcal{B}\|_\infty = \max_{i\in\{1,\dots,n\}} |\mathcal{B}_i|$. The width of an interval $\mathcal{A} \in \mathbb{R}^n$ is denoted by $\text{wd}(\mathcal{A}) \coloneqq \overline{\mathcal{A}} - \underline{\mathcal{A}}$. We extend the definitions [39] of arithmetic operations, set inclusion, and intersections of intervals to interval vectors and matrices componentwise.

Let a system characterized by $x \coloneqq [x_1, x_2]^\top \in \mathbb{R}^{2n}$, $x_i \coloneqq [x_{i_1}, \dots, x_{i_n}]^\top \in \mathbb{R}^n$, $i \in \{1, 2\}$, with dynamics

$$\dot{x}_1 = x_2, \tag{2a}$$
$$\dot{x}_2 = f(x) + g(x)u \tag{2b}$$

where $f \coloneqq [f_1, \dots, f_n]^\top : \mathbb{R}^{2n} \to \mathbb{R}^n$, $g \coloneqq [g_{ij}] : \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are *unknown*, locally Lipschitz functions, and $u \coloneqq [u_1, \dots, u_m]^\top \in \mathbb{R}^m$ is the control input.

The problem we consider is the invariance of $x_1(t)$ in a given closed set $\mathcal{C} \subset \mathbb{R}^n$ of the form

$$\mathcal{C} \coloneqq \{x_1 \in \mathbb{R}^n : h(x_1) \geq 0\}, \tag{3}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, with bounded derivative $\frac{\mathrm{d}h(\mathsf{x})}{\mathrm{d}\mathsf{x}}$ in $\text{Int}(\mathcal{C})$. More specifically, we aim to design a control law that achieves $x_1(t) \in \text{Int}(\mathcal{C})$, i.e., $h(x_1(t)) > 0$, for all $t \geq t_0$, given that $x_1(t_0) \in \text{Int}(\mathcal{C})$ for an initial time instant $t_0 \geq 0$.

As mentioned in Section 1, we aim to integrate a nonlinear feedback control scheme with a data-driven algorithm that approximates the dynamics (2) by using data obtained on the fly from a finite-horizon trajectory. More specifically, let a sequence $\{t_0, t_1, \dots\}$ of increasing time instants t_i, with $\Delta t_i \coloneqq t_{i+1} - t_i$, $i \in \bar{\mathbb{N}}$. At each t_i, the system obtains the measurement of $(x^i, \dot{x}^i, u^i)$, with

$$x^i \coloneqq [x^i_{1_1}, \dots, x^i_{1_n}, x^i_{2_1}, \dots, x^i_{2_n}]^\top \coloneqq x(t_i)$$
$$\dot{x}^i \coloneqq [\dot{x}^i_{1_1}, \dots, \dot{x}^i_{1_n}, \dot{x}^i_{2_1}, \dots, \dot{x}^i_{2_n}]^\top \coloneqq \dot{x}(t_i)$$
$$u^i \coloneqq [u^i_1, \dots, u^i_m]^\top \coloneqq u(t_i),$$

from the current trajectory of (2). Therefore, at each t_i, $i \in \bar{\mathbb{N}}$, the system has access to the discrete dataset $\mathscr{T}_i \coloneqq \{(x^j, \dot{x}^j, u^j)\}_{j=0}^i$. The trajectory that produces the dataset $\mathscr{T}_i$ has finite horizon in the sense that, for each finite i, $\mathscr{T}_i$ is finite. We are now ready to define the problem treated in this paper.

Problem 1 *Let a system evolve subject to the unknown dynamics (2), with $x_1(t_0) \in \text{Int}(\mathcal{C})$. Let a discrete sequence $\{t_i\}_{i\in\bar{\mathbb{N}}}$ of increasing time instants, and assume that the system obtains a measurement of $(x^i, \dot{x}^i, u^i)$ at each t_i, $i \in \bar{\mathbb{N}}$. Compute a time-varying, feedback-control law $u : \mathbb{R}^{2n} \times [t_0, \infty) \to \mathbb{R}^m$ that guarantees $x_1(t) \in \text{Int}(\mathcal{C})$, for all $t \geq t_0$.*

We further impose the following assumptions, required for the solution of Problem 1, where we use $\bar{\mathcal{A}} \coloneqq \mathcal{C} \times \mathbb{V}$, with $\mathbb{V} \subset \mathbb{R}^n$ being a compact set.

Assumption 1 *It holds that $\mathcal{C} \subset \mathcal{B}_r(0)$ for some $r > 0$.*

Assumption 2 *There exist known positive constants $\bar{f}_k$, $\bar{g}_{k\ell}$ satisfying $|f_k(\mathsf{x}) - f_k(\mathsf{y})| \leq \bar{f}_k|\mathsf{x} - \mathsf{y}|$, $|g_{k\ell}(\mathsf{x}) - g_{k\ell}(\mathsf{y})| \leq \bar{g}_{k\ell}|\mathsf{x} - \mathsf{y}|$, for all $k \in \{1, \dots, n\}$, $\ell \in \{1, \dots, m\}$, $\mathsf{x}, \mathsf{y} \in \bar{\mathcal{A}}$.*

Assumption 3 *There exist positive ν_h, ε_h such that $\|\nabla h(x_1)\| \geq \varepsilon_h$ for all x_1 satisfying $h(x_1) \in (0, \nu_h]$.*

Assumption 1 simply states that the system trajectory remains bounded when it evolves inside the safe set $\mathcal{C}$. Assumption 2 considers knowledge of upper bounds of the Lipschitz constants of f_k and g_{kl} in the safe set $\bar{\mathcal{A}}$. Note that we *do not assume* that the system is bounded in any set or exact knowledge of the Lipschitz constants. Assumption 3 is a simple controllability condition stating that $\dot{h}$ is not identically zero close to $\partial\mathcal{C}$.

Note that the problem setting exhibits unique challenges due to the on-the-fly availability of the data, the nonlinear nature of the dynamics, and the safety constraint $x_1(t) \in \mathcal{C}$. In contrast to most related works, we do not assume global boundedness, global Lipschitzness, or growth conditions on the dynamic terms [3–5], and we do not employ approximation structures or offline data offline [6, 9, 10]. Moreover, the safety constraint $x(t) \in \mathcal{C}$ introduces additional difficulties with respect to related works [25–29, 40, 41]: firstly, it imposes a requirement both on the transient- and the steady-state trajectory of the system, unlike the asymptotic property of stability. Secondly, the allowed fraction of the state space that the system can explore and obtain measurements in is restricted by $\mathcal{C}$. The solution of Problem 1, consisting of a two-layered approach, is given in Sections 3-5. Firstly, we use previous results on on the fly approximation of the unknown dynamics [4] and compute Lipschitz estimates for $g(x)$. Secondly, we use these estimates to design a feedback controller based on reciprocal barriers.

Algorithm 1 $\mathsf{Approximate}(t_i, \mathscr{T}_i)$

Input: Single trajectory $\mathscr{T}_i$, sufficiently large $M > 0$.
Output: $\mathscr{E}_i = \{(x^j, C^j_{\mathcal{F}}, C^j_{\mathcal{G}}) | f(x^j) \in C^j_{\mathcal{F}}, g(x^j) \in C^j_{\mathcal{G}}\}_{j=0}^{i}$

1: $\mathcal{R}^{f_{\mathcal{A}}} \leftarrow [-M, M]^n$, $\mathcal{R}^{G_{\mathcal{A}}} \leftarrow [-M, M]^{n\times m}$
2: $C^0_{\mathcal{F}} \leftarrow \mathcal{R}^{f_{\mathcal{A}}}$
3: $C^0_{\mathcal{G}} \leftarrow \mathcal{R}^{G_{\mathcal{A}}}$
4: $\mathscr{E}_0 \leftarrow \{(x^0, C^0_{\mathcal{F}}, C^0_{\mathcal{G}})\}$
5: **for** $\iota \in \{1, \ldots, i\} \wedge (x^\iota, \dot{x}^\iota, u^\iota) \in \mathscr{T}_i$ **do**
6: $\quad \mathcal{F}^\iota \leftarrow \boldsymbol{F}(x^\iota, \mathscr{E}_{\iota-1})$,
7: $\quad \mathcal{G}^\iota \leftarrow \boldsymbol{G}(x^\iota, \mathscr{E}_{\iota-1})$
8: $\quad (C^\iota_{\mathcal{F}}, C^\iota_{\mathcal{G}}) \leftarrow \mathsf{UpdateSets}(x^\iota, \dot{x}^\iota, u^\iota, \mathcal{F}^\iota, \mathcal{G}^\iota)$
9: $\quad \mathscr{E}_\iota \leftarrow \mathscr{E}_{\iota-1} \cup \{(x^\iota, C^\iota_{\mathcal{F}}, C^\iota_{\mathcal{G}})\}$
10: **do**
11: $\quad$ Execute lines 5–9 with $\mathscr{E}_i$ instead of $\mathscr{E}_{\iota-1}$ in line 6
12: **while** $\mathscr{E}_i$ is not invariant
13: **return** $\mathscr{E}_i$

3 On-the-fly over-approximation of the dynamics

In this section, we use the approximation algorithm of [4], which is based on data obtained online from a single finite-horizon trajectory, to compute an estimate of the control matrix $g(x)$. The approximation algorithm uses, at each t_i, $i \in \bar{\mathbb{N}}$, the information from the finite dataset $\mathscr{T}_i$ in order to construct a data-driven differential inclusion $\dot{x} \in \mathcal{F}^i(x) + \mathcal{G}^i(x)u$ that contains the unknown vector fields of (2), where $\mathcal{F}^i : \mathbb{R}^{2n} \to \mathbb{IR}^n$ and $\mathcal{G}^i : \mathbb{R}^{2n} \to \mathbb{IR}^{n\times m}$ are known interval-valued functions. Such an over-approximation enables us to provide a locally Lipschitz estimate $\hat{g}^i$ of g to be used in the subsequent feedback control scheme.

The over-approximation algorithm is illustrated in Algorithm 2. The functions $\boldsymbol{F} := (\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n) \in \mathbb{IR}^n$ and $\boldsymbol{G} := (\boldsymbol{G}_{k\ell}) \in \mathbb{IR}^{n\times m}$, used in lines 6 and 7, respectively, are given by

$$\boldsymbol{F}_k(x, \mathscr{E}_i) := \bigcap_{(x^j, C^j_{\mathcal{F}}, \cdot) \in \mathscr{E}_i} \left\{ C^j_{\mathcal{F}_k} + \bar{f}_k \|x - x^j\| [-1, 1] \right\} \tag{4a}$$

$$\boldsymbol{G}_{k\ell}(x, \mathscr{E}_i) := \bigcap_{(x^j, \cdot, C^j_{\mathcal{G}}) \in \mathscr{E}_i} \left\{ C^j_{\mathcal{G}_{k\ell}} + \bar{g}_{k\ell} \|x - x^j\| [-1, 1] \right\} \tag{4b}$$

for all $k \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, m\}$, where the set $\mathscr{E}_i$ has the form $\mathscr{E} := \{(x^j, C^j_{\mathcal{F}}, C^j_{\mathcal{G}})\}_{j=0}^{i-1}$, for some $i \in \bar{\mathbb{N}}$.

The function $\mathsf{UpdateSets}()$, used in line 8, updates the sets $C^\iota_{\mathcal{F}} := (C^\iota_{\mathcal{F}_1}, \ldots, C^\iota_{\mathcal{F}_n})$and $C^\iota_{\mathcal{G}} := (C^\iota_{\mathcal{G}_{k\ell}})$ as

$$C^\iota_{\mathcal{F}_k} := \{\mathcal{F}^\iota_k\} \cap \{\dot{x}^\iota_k - \mathcal{Y}^\iota_k\}, \tag{5a}$$

$$s_{0,k} := \{\dot{x}^\iota_k - C^\iota_{\mathcal{F}_k}\} \bigcap \{\mathcal{Y}^\iota_k\}, \tag{5b}$$

$$C^\iota_{\mathcal{G}_{k\ell}} := \begin{cases} \left(\left\{s_{\ell-1,k} - \sum_{p>\ell} \mathcal{G}^\iota_{kp} u^\iota_p\right\} \bigcap \{\mathcal{G}^\iota_{k\ell} u^\iota_\ell\}\right) \frac{1}{u^\iota_\ell}, u^\iota_\ell \neq 0 \\ \mathcal{G}^\iota_{k\ell}, \quad \text{otherwise}, \end{cases} \tag{5c}$$

$$s_{\ell,k} := \{s_{\ell-1,k} - C^\iota_{\mathcal{G}_{k\ell}} u^\iota_\ell\} \bigcap \left\{ \sum_{p>\ell} \mathcal{G}^\iota_{kp} u^\iota_p \right\}, \tag{5d}$$

for all $k \in \{1, \ldots, n\}$, $\ell \in \{1, \ldots, m\}$, and some $\iota \in \{1, \ldots, i\}$, where $\mathcal{Y}^\iota := (\mathcal{Y}^\iota_1, \ldots, \mathcal{Y}^\iota_n) := \mathcal{G}^\iota u^\iota \in \mathbb{IR}^n$.

We briefly explain the intuition behind Algorithm 2. The algorithm refines iteratively the sets $C^\iota_{\mathcal{F}}$, $C^\iota_{\mathcal{G}}$, $\mathcal{F}^\iota$, and $\mathcal{G}^\iota$ to over-approximate f and g at each data point of $\mathscr{J}_i$. More specifically, it updates the intervals $\mathcal{F}^\iota$ and $\mathcal{G}^\iota$ in (4) using the Lipschitz estimates $\bar{f}_k$, $\bar{g}_{k\ell}$, and the sets $C^\iota_{\mathcal{F}}$ and $C^\iota_{\mathcal{G}}$ in (5) using the dynamics constraint (2). In fact, it can be proven that $f(x) \in \boldsymbol{F}(x, \mathscr{E}_i)$ and $g(x) \in \boldsymbol{G}(x, \mathscr{E}_i)$, for all $x \in \bar{\mathcal{A}}$, $i \in \bar{\mathbb{N}}$, and that $C^\iota_{\mathcal{F}}$, $C^\iota_{\mathcal{G}}$ are the smallest intervals enclosing $f(x^\iota)$ and $g(x^\iota)$, respectively, given only $(x^\iota, \dot{x}^\iota, u^\iota)$, $\mathcal{F}^\iota$ and $\mathcal{G}^\iota$ [4]. The computational complexity of the algorithm (in time and memory) is quadratic in the number of elements of $\mathscr{T}_i$ and linear in the system dimension n.

The subsequent theorem characterizes the correctness of the obtained differential inclusions.

Theorem 1 ([4], Theorem 1) *Let $i \in \mathbb{N}$ and $\mathsf{F}^i := (\mathsf{F}^i_1, \ldots, \mathsf{F}^i_n) : \mathbb{R}^{2n} \to \mathbb{IR}^n$, $\mathsf{G}^i := (\mathsf{G}^i_{k\ell}) : \mathbb{R}^{2n} \to \mathbb{IR}^{n\times m}$, with $\mathsf{F}^i(x) := \boldsymbol{F}(x, \mathscr{E}_i)$, $\mathsf{G}^i(x, \mathscr{E}_i) := \boldsymbol{G}(x)$ where $\mathscr{E}_i$ is output of Algorithm 1, which is executed at t_i using the dataset $\mathscr{T}_i$. Then it holds that*

$$\dot{x}(t) \in \mathsf{F}^i(x(t)) + \mathsf{G}^i(x(t))u, \forall t \geq t_i.$$

Remark 1 *As pointed out in [4], we can adjust Algorithm 1 to account for extra information on f and g, if available, yielding tighter approximations F^i and G^i. In particular, if we are given sets $\mathcal{A} \subseteq \bar{\mathcal{A}}$, $\mathcal{R}^{f_{\mathcal{A}}}$, $\mathcal{R}^{G_{\mathcal{A}}}$ such that $\{f(x) | x \in \mathcal{A}\} \subseteq \mathcal{R}^{f_{\mathcal{A}}}$ and $\{g(x) | x \in \mathcal{A}\} \subseteq \mathcal{R}^{G_{\mathcal{A}}}$, these can be used in Algorithm 1, replacing the respective ones defined in line 1. We stress, nevertheless, that we do not require the availability of such sets.*

Based on Theorem 1, we propose now a locally Lipschitz function $\hat{g}^i : \bar{\mathcal{A}} \to \mathbb{R}^{n\times m}$ that estimates the unknown function g at each time instant t_i.

Lemma 1 *Let $i \in \bar{\mathbb{N}}$. Given $\theta \in [0, 1]$, each component of the function $\hat{g}^i := [\hat{g}^i_{k\ell}] : \bar{\mathcal{A}} \to \mathbb{R}^{n\times m}$ given by*

$$\hat{g}^i_{k\ell}(x) = \theta \underline{\mathsf{G}}^i_{k\ell}(x) + (1 - \theta) \bar{\mathsf{G}}^i_{k\ell}(x), \tag{6}$$

where $\underline{\mathsf{G}}^i_{k\ell}$ and $\bar{\mathsf{G}}^i_{k\ell}$ are the left and right endpoints, respectively, of the interval $\mathsf{G}^i_{k\ell}$, is locally Lipschitz in $\bar{\mathcal{A}}$, for all $k \in \{1,\dots,n\}$ and $\ell \in \{1,\dots,m\}$.

Remark 2 (Measurement noise) *We can extend the proposed methodology to account for noise in the obtained measurements related to $\dot{x}$. In particular, we can consider cases where the systems obtains, at each t_i, $i \in \bar{\mathbb{N}}$, the measurement $(x^i, \widetilde{\dot{x}^i}, u^i)$, where $\widetilde{\dot{x}^i} := \dot{x}^i + \Delta\dot{x}^i$, and $\Delta\dot{x}_i \in \mathbb{R}^{2n}$ is bounded measurement noise. Algorithm 2 can then incorporate in (5) an estimate of the upper bound of $\Delta\dot{x}^i$ to update $C^i_{\mathcal{F}_k}$ and $s_{0,k}$, leading, nevertheless, to more conservative approximations F^i and G^i. More details can be found in [42].*

4 Control design and Safety Guarantees

This section presents the main results of the paper. We first propose, in Section 4.1, a learning-based control algorithm that relies on the approximation of the dynamics of Section 3 and the concept of reciprocal barriers. Next, we provide in Section 4.2 bounds on the approximation errors $\hat{g}^i_{kl}(x^i) - g_{kl}(x^i)$, $k \in \{1,\dots,n\}$, $\ell \in \{1,\dots,m\}$, $i \in \bar{\mathbb{N}}$, based on the frequency of the update time instants t_i, $i \in \mathbb{N}$. Finally, Section 4.3 presents a simplified version of the algorithm for the special case where $g(x)$ is square and $g(x)+g(x)^\top$ is positive definite.

4.1 Learning-based Control Design

Given the set $\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}$ and following [15], we define a continuously differentiable *reciprocal* barrier function $\beta : (0,\infty) \to \mathbb{R}$ that satisfies

$$\frac{1}{\alpha_1(h)} \leq \beta(h) \leq \frac{1}{\alpha_2(h)} \tag{7}$$

for class $\mathcal{K}$ functions α_1, α_2. Note that (7) implies $\inf_{x\in \mathrm{Int}(\mathcal{C})} \beta(h(x)) > 0$ and $\lim_{x\to\partial\mathcal{C}} \beta(h(x)) = \infty$. In order to render $\mathcal{C}$ forward-invariant, we aim to design a control algorithm that guarantees the boundedness of β in a compact set. We further require the extra condition:

$$\max\left\{\left\|\frac{\mathrm{d}\beta(h)}{\mathrm{d}h}\right\|, \left\|\frac{\mathrm{d}^2\beta(h)}{\mathrm{d}h^2}\right\|\right\} \leq \frac{1}{\alpha_4(h)}, \tag{8}$$

for a class $\mathcal{K}$ function α_4, implying that the derivatives of β are bounded when x_1 lies in compact subsets of $\mathrm{Int}(\mathcal{C})$. Examples of β include $\beta(h) = \frac{1}{h}$, $\beta(h) = -\ln\left(\frac{h}{1+h}\right)$. We further define

$$\beta_{\mathrm{d}} := \beta_{\mathrm{d}}(x_1) := \frac{\mathrm{d}\beta(h(x_1))}{\mathrm{d}h(x_1)}.$$

Ideally, we would like the system to deviate minimally from a potential nominal task assigned to it, dictated by a nominal continuous controller $u_{\mathrm{n}}(x)$. Therefore, as in [15], we would like to allow β to grow when x_1 is not close to the boundary of $\mathcal{C}$. To this end, we use a *decreasing* continuous signal $\sigma_\mu : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying $\sigma_\mu(\mathsf{x}) = 1$ for $\mathsf{x} \leq 0$ and $\sigma_\mu(\mathsf{x}) = 0$ for $\mathsf{x} \geq \mu$, for some $\mu > 0$. For a given constant $\mu_x > 0$, we want the control law to act only when $0 < h(x_1) \leq \mu_x$, which defines the set

$$\mathcal{C}_{\mu_x} := \{x_1 \in \mathbb{R}^n : h(x_1) \in (0, \mu_x]\}.$$

Therefore, following a backstepping-like scheme, we design first a reference signal for x_2 as

$$x_{2,\mathrm{r}} := x_{2,\mathrm{r}}(x_1) := -\kappa_x \sigma_{\mu_x}(h(x_1))\beta_{\mathrm{d}}(x_1)\nabla h(x_1), \tag{9}$$

where κ_x is a constant control gain, and define the error

$$e_2 := e_2(x) := x_2 - x_{2,\mathrm{r}} \tag{10}$$

We choose the constant μ_x small enough such that $\mu_x < \nu_h$ implying $\|\nabla h(x_1)\| \geq \varepsilon_h$ for all $x_1 \in \mathcal{C}_{\mu_x}$, according to Assumption 3.

We next design the control law such that β and e_2 remain bounded. To that end, we define a continuously differentiable function $h_v : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying $h_v(x) > 0$ if and only if $\|e_2\| < \bar{B}_2$, for a positive $\bar{B}_2$, and define

$$\mathcal{C}_v := \{x \in \mathbb{R}^{2n} : h_v(x) \geq 0\} \subset \mathbb{R}^{2n}.$$

An example is ellipsoidal functions of the form $h_v(x) := 1 - e_2(x)^\top A_v e_2(x)$ for an appropriate $A_v \in \mathbb{R}^{n\times n}$. The function h_v depends explicitly on e_2, i.e., the difference $x_2 - x_{2,\mathrm{r}}(x_1)$, but we define it as a function of x to ease the subsequent analysis. We choose the constant κ_x in (9) and h_v such that $\mathcal{C} \cap \mathcal{C}_v \subset \bar{\mathcal{A}}$ (see Assumption 2), so that Algorithm 1 produces valid approximations of the dynamic terms. We also choose the constant $\bar{B}_2$ such that $h_v(x(t_0)) > 0$. Next, we design the control law to guarantee $h_v(x(t)) > 0$, for all $t \geq t_0$. Following similar steps as with $h(x_1)$, we define a continuously differentiable reciprocal function $\beta_v : (0,\infty) \to \mathbb{R}$ that satisfies (7) for for class $\mathcal{K}$ functions α_{v_1}, α_{v_2}, and which we aim to maintain bounded. As in the case of β, we define

$$\beta_{v,\mathrm{d}} := \beta_{v,\mathrm{d}}(x) := \frac{\mathrm{d}\beta_v(h_v(x))}{\mathrm{d}h_v(x)}.$$

Similar to the definition of $\mathcal{C}_{\mu_x}$, we choose a constant $\mu_v > 0$ that defines the set

$$\mathcal{C}_{v,\mu_v} := \{x \in \mathbb{R}^{2n} : h_v(x) \in (0, \mu_v]\}, \tag{11}$$

where we aim to enable the safety control law.

Let now the estimate $\hat{g}^i(x)$ of $g(x)$, as computed by Lemma 1, for $t \in [t_i, t_{i+1})$, $i \in \bar{\mathbb{N}}$. Let also the error

$$\widetilde{g}^i(x) := \hat{g}^i(x) - g(x)$$

for $i \in \bar{\mathbb{N}}$. We use the term $\hat{g}^i(x)$ in the control design to cancel the effect of $g(x)$, inducing thus a time-dependent switching. More specifically, we set u as

$$u(x,t) := u_{\mathrm{n}}(x) - \kappa_v \sigma_{\mu_v}(h_v(x))\beta_{v,\mathrm{d}}(x)\frac{\hat{g}^i(x)^\top \nabla h_v(x)}{\|\hat{g}^i(x)^\top \nabla h_v(x)\|^2}, \tag{12}$$

for $t \in [t_i, t_{i+1})$, $i \in \bar{\mathbb{N}}$, where $u_{\mathrm{n}}(x)$ is a nominal continuous controller, and κ_v is a positive constant.

Note that the proposed control scheme, consisting of Algorithm 2 and (9)-(12), does not use any a priori information of $f(x)$ and $g(x)$; it estimates $g(x)$ by using the data obtained on the fly, and does not use any approximation regarding $f(x)$. Further, the algorithm does not use the instants $t_0, t_1, \dots,$ or the rate $\Delta t_i = t_{i+1} - t_i$ and hence these do not need to be given a priori.

Before stating the main result of this section, we establish the existence of solutions of the closed-loop system.

Lemma 2 *Let a system evolve according to the dynamics* (2) *and control scheme* (9)-(12)*; Let also a set* $\mathcal{C}$ *satisfying* $x_1(t_0) \in \mathrm{Int}(\mathcal{C})$ *for some* $t_0 \geq 0$*. Then there exists a unique, maximal solution* $x : [t_0, t_{\max}) \to \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$ *for some* $t_{\max} > t_0$*, where* $\bar{\mathcal{C}} := \{x \in \mathbb{R}^{2n} : x_1 \in \mathcal{C}\}$.

We are now ready to state the main results of this paper. More specifically, the next theorem states that, if the estimated system defined by $\hat{g}^i(x)$ is controllable with respect to $h_v(x)$, and if $\hat{g}^i(x)$ is sufficiently close to $g(x)$, we achieve boundedness of $x_1(t)$ in $\mathrm{Int}(\mathcal{C})$ and, consequently, provide a solution to Problem 1.

Theorem 2 *Let a system evolve according to* (2) *and* (9)-(12)*. Let also a set* $\mathcal{C}$ *satisfying* $x_1(t_0) \in \mathrm{Int}(\mathcal{C})$ *for some* $t_0 \geq 0$*. Define, for* $\mu > 0$*,*

$$T_\mu := \{t \in [t_0, t_{\max}) : x(t) \in \mathrm{Int}(\mathcal{C}_{v,\mu})\}$$

and $i_t := \max\{i \in \bar{\mathbb{N}} : t_i \leq t\}$, $t \in [t_0, t_{\max})$*. Assume there exists* $\mu'_v \in (0, \mu_v)$ *such that the following holds: for each* $t \in T_{\mu'_v}$*, there exist* $r := r(t, \mu'_v) > 0$*,* ε_1*,* ε_2*, with* $\varepsilon_1 > \varepsilon_2$*, for which the following conditions hold*[1]

$$\mathcal{B}_r(x(t)) \subset \mathcal{C}_v \tag{13a}$$

$$\|\hat{g}^{i_t}(y)^\top \nabla_{x_2} h_v(y)\| \geq \varepsilon_1, \tag{13b}$$

$$\|\widetilde{g}^{i_t}(y)\|\|\nabla_{x_2} h_v(y)\| < \varepsilon_2 \sigma_{\mu_v}(\mu'_v), \tag{13c}$$

for all $y \in \mathcal{B}_r(x(t))$*. Then, under Assumptions 1 and 3, it holds that* $x_1(t) \in \mathrm{Int}(\mathcal{C})$*, and all closed-loop signals remain bounded, for all* $t \geq t_0$*.*

[1] $\nabla_{x_2} h_v(y)$: means the gradient is evaluated at the point y.

We now elaborate on the required conditions (13), which concern the trajectory of the system close to $\partial\mathcal{C}_v$.

Firstly, (13b) is a sufficient controllability condition that allows the boundedness of the control law (12) close to $\partial\mathcal{C}_v$. Note that, unlike the majority of the related works, which assume the structural condition of square and positive definite $g(x)$, (13b) depends on the estimate $\hat{g}$ of g as well as the choice of $h_v(x)$. We exploit this dependency in Section 5 to relax condition (13b) via an online mechanism that locally switches to a different $h_v(x)$.

Secondly, condition (13c) implies that the system will have computed a sufficiently good approximation of $g(x)$, via $g^i(x)$, before reaching the boundary of $\mathcal{C}_v$. Such an approximation can be practically achieved via rich exploration of the state space and large frequency of measurements $\{(x^i, \dot{x}^i, u^i)\}$. In turn, one can accomplish the former by selecting a persistently exciting nominal controller u_{n}, such as a high-frequency sinusoidal signal with small amplitude. However, the explicit derivation of a condition on the persistence of excitation of the nominal controller is out of the scope of this paper. Moreover, note that (13c) is only *sufficient* and not necessary; the proposed algorithm might maintain the safety of the system even if (13c) does not hold. Regarding the frequency of measurement updates, we provide in Section 4.2 a relation among $t_{i+1} - t_i$ and $\|\widetilde{g}^i(x^{i+1})\|$.

4.2 Bounds on $\widetilde{g}^i(x)$

Theorem 2 is based on a small enough error of the approximation error $\widetilde{g}^i(x)$. As explained in Section 4.1, this is achieved through sufficient exploration of the state space and a sufficiently high frequency of the measurement updates t_i, $i \in \bar{\mathbb{N}}$. In this section, we provide a closed-form relation between the approximation error $\widetilde{g}^i(x^{i+1})$ and the difference $\Delta t_i = t_{i+1} - t_i$.

We begin by defining some preliminary notation. Note first that Theorem 2 proves the boundedness of the control signal $u(x,t)$; we use $\mathcal{U}^i \in \mathrm{I\!R}^m$ to denote the bounded set satisfying $u(x(t),t) \in \mathcal{U}^i \in \mathrm{I\!R}^m$, for all $t \in [t_i, t_{i+1}]$ for $i \in \bar{\mathbb{N}}$. Moreover, we define the terms

$$\boldsymbol{h}(x,u) := \boldsymbol{F}(x) + \boldsymbol{G}(x)u \tag{14a}$$

$$\delta^i := \sqrt{\sum_{k=1}^{n}(\bar{f}_k + \sum_{\ell=1}^{m}\bar{g}_{k\ell}|\mathcal{U}^i_\ell|)^2} \tag{14b}$$

$$\mathcal{S}^i := x^i + \frac{\Delta t\|\boldsymbol{h}(x^i,\mathcal{U}^i)\|_\infty}{1 - \sqrt{n}\Delta t_i \delta^i}[-1,1]^n \tag{14c}$$

$$\mathcal{K}^i := (\mathcal{J}^f + \mathcal{J}^g\mathcal{U}^i)\Big(\boldsymbol{h}(x^i,\mathcal{U}^i) + \frac{\Delta t_i\|\boldsymbol{h}(x^i,\mathcal{U}^i)\|_\infty}{1 - \sqrt{n}\Delta t_i \delta^i}\mathcal{H}^i\Big) \tag{14d}$$

$$\mathcal{H}^i := (\bar{f} + \bar{g}\mathcal{U})[-\sqrt{n}, \sqrt{n}]^n, \tag{14e}$$

for $i \in \bar{\mathbb{N}}$, where $\mathcal{J}^f := (\mathcal{J}^f_{kp}) \in \mathrm{I\!R}^{n\times n}$ and $\mathcal{J}^g := (\mathcal{J}^g_{k\ell p}) \in \mathrm{I\!R}^{n\times m\times n}$ denote the over-approximations of the Jacobian of f and g, respectively, and are given by $\mathcal{J}^f_{kp} = \bar{f}_k[-1,1]$ and $\mathcal{J}^g_{k\ell p} = \bar{g}_{k\ell}[-1,1]$ for all $k,p \in \{1,\dots,n\}$ and $\ell \in \{1,\dots,m\}$. We are now ready to provide a relation between $\widetilde{g}^i(x^{i+1})$ and Δt_i.

Lemma 3 (Point-based estimation error) *Let the current state $x^i = x(t_i)$ and the bounded admissible set of control values between time t_i and $t_{i+1} = t_i + \Delta t_i$ be $\mathcal{U}^i \in \mathbb{IR}^m$, $i \in \bar{\mathbb{N}}$. Under the assumption that $(\sqrt{n}\delta^i)\Delta t < 1$, it holds that*

$$\|\hat{g}^i_{k\ell^\cdot}(x^{i+1}) - g_{k\ell}(x^{i+1})\| \leq \mathrm{wd}(C^i_{\mathcal{G}_{k\ell}}) + 2\bar{g}_{k\ell}\overline{\|\boldsymbol{h}(x^i,\mathcal{U}^i)\|}\Delta t_i + 2\bar{g}_{k\ell}\overline{\|\mathcal{K}^i\|}\frac{\Delta t_i^2}{2} \quad (15)$$

for all $k \in \{1,\dots,n\}$, and $\ell \in \{1,\dots,m\}$.

Relation (15) connects the approximation error $\widetilde{g}^i(x^{i+1})$ and the intervals Δt_i, implying that more frequent measurements (smaller Δt_i) might lead to smaller $\widetilde{g}^i(x^{i+1})$. The term $\mathrm{wd}(C^i_{\mathcal{G}_{k\ell}})$ is independent of Δt_i and depends on the richness of obtained data (see Algorithm 1).

4.3 Square Control Matrix $g(x)$

The fact that g is non-square and completely unknown and hence cannot be used in the control algorithm makes the considered problem significantly more challenging compared to other works in the related literature that assume positive definiteness of g or of $g + g^\top$ [6–8, 24]. In fact, we show now that a simple feedback control law can solve Problem 1 in the case of square and positive definite matrix g, without using measurements on $\dot{x}$. We first need Assumption 3 to hold for $\nabla_{x_2} h_2(x)$:

Assumption 4 *There exist positive ν_v, ε_v such that $\|\nabla_{x_2} h_v(x)\| \geq \varepsilon_v$ for all x satisfying $h_v(x) \in (0, \nu_v]$.*

As with (12), we select a positive constant μ_v to enable the control law in the set $\mathcal{C}_{v,\mu_v} = \{x \in \mathbb{R}^{2n} : h_v(x) \in (0, \mu_v]\}$. Similarly to μ_x, we choose the constant μ_v sufficiently small so that it satisfies $\mu_v < \nu_v$, implying $\|\nabla_{x_2} h_v(x)\| \geq \varepsilon_v$ for all $x \in \mathcal{C}_{\mu_v}$.

Given the reference signal $x_{2,\mathrm{r}}$ in (9), the function $h_v(\cdot)$ and $\beta_v(\cdot)$, the switching function $\sigma_\mu()$, and the constant μ_v, we design now the control law as

$$u \coloneqq u(x,t) \coloneqq u_{\mathrm{n}}(x) - \kappa_v \sigma_{\mu_v}(h_v(x))\beta_{v,\mathrm{d}}(x)\nabla_{x_2} h_v(x), \quad (16)$$

whose correctness is proven in the following theorem.

Theorem 3 *Let a system evolve according to the dynamics (1) and control law (16), with $m = n$ and let a set $\mathcal{C}$ satisfying $x_1(t_0) \in \mathrm{Int}(\mathcal{C})$ for some $t_0 \geq 0$. Assume there exists a constant $\mu'_v \in (0, \mu)$ such that $\lambda_{\min}\left(g(x) + g(x)^\top\right) > 0$, for all $x \in \mathcal{C}_{v,\mu'_v}$. Under Assumptions 1, 3, it holds that $x_1(t) \in \mathrm{Int}(\mathcal{C})$, and all closed-loop signals are bounded, for all $t \geq t_0$.*

Note that, if $g(x)$ is square and satisfies $\lambda_{\min}(g(x) + g(x)^\top)$, there is no need for its approximation through online data and the respective assumption on the approximation error (13b) is no longer required. Intuitively, $g(x)$ retains the direction of the applied control input, which is sufficient to guarantee safety.

Remark 3 (High relative degree) *Theorems 2 and 3 suggest a way to tackle higher-order dynamics of the form*

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1},\ i \in \{1,\dots,k-1\}, \\ \dot{x}_k &= f(x_k) + g(x)u \end{aligned}$$

for a positive integer k, where $\bar{x}_i \coloneqq [x_1^\top,\dots,x_i^\top]^\top \in \mathbb{R}^{n\cdot i}$, for all $i \in \{1,\dots,k\}$, and $x \coloneqq \bar{x}_k$. By assuming that $g_i + g_i^\top$ are positive definite, for all $i \in \{1,\dots,k-1\}$, we can design continuous reference signals $x_{i+1,\mathrm{r}}$ for the states x_{i+1}, as in (9) and based on consecutive error signals as in (10). The control signal u can then be designed based on the over-approximation of $g(x)$ and a reciprocal barrier on the difference $x_k - x_{k,\mathrm{r}}$, as in (12).

5 Controllability loss

In this section we provide an algorithm that considers cases where $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\|$ can become arbitrarily small, relaxing thus the condition (13b) in Theorem 2. Note that, since we desire $\hat{g}^i(x)$ to be close to $g(x)$, and hence the interval $\mathsf{G}^i(x)$ to be small (see Lemma 1), choosing a different $\hat{g}^i(x)$ from $\mathsf{G}^i(x)$ is not expected to significantly modify the term $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\|$ and resolve the controllability issue at hand.

The framework we follow in order to tackle such cases is an online switching mechanism that computes locally alternative barrier functions h_j, defining new safe sets

$$\mathcal{C}_j \coloneqq \{x \in \mathbb{R}^n : h_j(x) \geq 0\} \subset \mathcal{C}_{j-1},$$

for $j \geq 2$, and $h_1 \coloneqq h_v$, $\mathcal{C}_1 \coloneqq \mathcal{C}_v$. More specifically, at a point x_c where $\|\hat{g}^i(x_c)^\top \nabla_{x_2} h_v(x_c)\|$ becomes too small, the algorithm looks for an alternative function h_2 with $\mathcal{C}_2 \subset \mathcal{C}_v$ and $\|\hat{g}^i(x_c)^\top \nabla_{x_2} h_2(x_c)\|$ is sufficiently large. The next example illustrates this reasoning.

Example 1 *Consider a system with $x_1 = [x_{11}, x_{12}]^\top \in \mathbb{R}^2$, $x_2 = [x_{21}, x_{22}^\top]^\top \in \mathbb{R}^2$, $\hat{g}^i(x) = [2x_{21}, -2]^\top$ for some $i \in \bar{\mathbb{N}}$, and $h_v(x) = 4.5 - \|x_2\|^2$, representing a sphere with radius of $\sqrt{4.5}$ (depicted with red in Fig. 2). The term $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\|$ vanishes on the parabola $x_{22} = x_{21}^2$ (depicted with black in Fig. 2). When x is close enough to that line, the proposed algorithm computes a new h_2;*

Fig. 2 depicts the case when $x_c = [1, 0.95]$ (green asterisk). A potential choice is then the ellipsoidal set $h_2 = -0.175x_{21}^2 + 0.2x_{21}x_{22} - 0.2x_{22}^2 + 0.15x_{21} - 0.0889x_{22} + 0.2$ (depicted with blue in Fig. 2). The new h_2 satisfies $C_2 \subset \mathcal{C}_v$, $h_2(x_c) > 0$, while $\|\hat{g}^i(x)^\top \nabla_{x_2} h_2(x)\|$ vanishes on the parabola $0.1x_{21} - 0.8x_{22} - 0.4x_{21}x_{22} + 0.7x_{21}^2 - 0.1779$ (depicted with purple in Fig. 2), with $\|\hat{g}^i(x_c)^\top \nabla_{x_2} h_2(x_c)\| = 1.6379$. The controller switches locally to h_2 until $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\|$ becomes sufficiently large again. In case the system navigates along the line $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\| = 0$ to a region where $\|\hat{g}^i(x)^\top \nabla_{x_2} h_2(x)\|$ is also small, the procedure is repeated and a new h_3 is computed.

In the aforementioned example, the region around the point where both $\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\| = 0$ and $\|\hat{g}^i(x)^\top \nabla_{x_2} h_2(x)\| = 0$ is around the intersection of the black and purple lines in Fig. 2. Note, however, that in the specific example, the system cannot navigate close to that point since employment of h_2 will keep it bounded in $\mathcal{C}_2$ (the set defined by the blue line in the figure).

Similarly to $h_v(x)$, the functions h_j, $j \geq 2$, depend explicitly on e_2, i.e., the difference $x_2 - x_{2,\mathrm{r}}(x_1)$. Moreover, for technical requirements, we assume in the following that $\mathcal{C}_v$ is a compact $2n$-dimensional manifold.

The aforementioned procedure is described more formally in the algorithm SafetyAdaptation (Algorithm 2). More specifically, for a given j, each ρ_j indicates whether the system is close to the set where $\|\hat{g}^i(x)^\top \nabla_{x_2} h_j(x)\| = 0$. If that's the case ($\rho_j = 0$), the algorithm computes a new function h_{j+1} such that $C_{j+1} \subset C_j$, and in the switching point it holds that $h_{j+1}(x) > 0$ and $\|\hat{g}^i(x)^\top \nabla_{x_2} h_{j+1}(x)\|$ is sufficiently large; the control law uses then $h_{j+1}(x)$. If $\|\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)\|$ becomes sufficiently large, for some $\iota < j$, the algorithm sets j back to ι, and the control uses $h_\iota(x)$ again. We also impose a hysteresis mechanism for the switching of the constants ρ_ι (lines 4, 10) through the parameters $\bar{\varepsilon}$, $\underline{\varepsilon}$.

The reasoning behind Algorithm 2 is the following. By appropriately choosing the functions $h_\iota(x)$, the solutions of $\|\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)\| = 0$ form curves of measure zero. Hence, the intersection of $2n$ such lines will be a single point in $\mathbb{R}^{2n}$ and hence the employment of a newly computed h_{2n+1} will drive the system away from that point, resetting the algorithm. More formally, there is no $t \geq t_0$ and $j \geq 2n+1$ such that $\|g^i(x(t))^\top \nabla_{x_2} h_j(x(t))\| \leq \underline{\varepsilon}$, implying that the iterator variable j of Algorithm 2 will never exceed $2n+1$.

Following similar steps as with h and h_v, we define continuously differentiable reciprocal functions $\beta_j : (0, \infty) \to \mathbb{R}$ that satisfy (7) for class $\mathcal{K}$ functions α_{j_1}, α_{j_2}, and $j \in \{1, \ldots, 2n+1\}$. The formal definition of the control law is then, for all $t \in [t_i, t_{i+1})$ and $i \in \bar{\mathbb{N}}$,

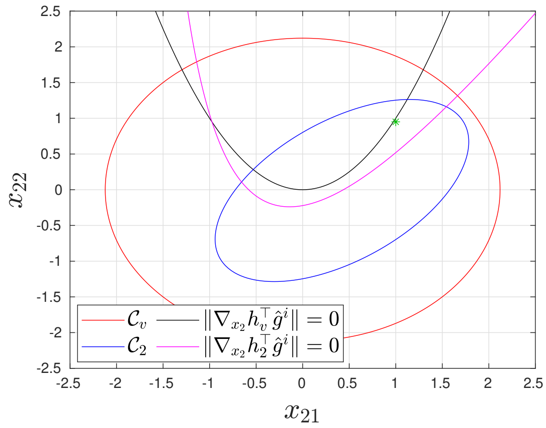

Fig. 2. Illustration of the adaptation algorithm (Algorithm 2). At a point x_c on the curve $\|g^i(x)^\top \nabla_{x_2} h_v(x)\| = 0$ (green point on black curve), the algorithm computes a new $h_2(x)$ such that $\mathcal{C}_2 \subset \mathcal{C}_v$ (blue curve) and x_c is sufficiently far from the curve $\|g^i(x)^\top \nabla_{x_2} h_2(x)\| = 0$ (purple curve).

$$u = u_{\mathrm{n}}(x) - \kappa_v \sigma_{\mu_v}(h_v(x)) u_{\mathrm{b}}(x,t) \tag{17a}$$

$$u_{\mathrm{b}} := \sum_{\iota=1}^{2n+1} \rho_\iota \prod_{j=1}^{\iota-1} (1-\rho_j) \beta_{j,\mathrm{d}} \frac{\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)}{\|\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)\|^2}, \tag{17b}$$

with $h_1 = h_v$, $\beta_1 = \beta_v$, and $\beta_{j,\mathrm{d}} := \frac{\mathrm{d}\beta_j(h_j(x))}{\mathrm{d}h_j(x)}$.

The SafetyAdaptation algorithm is run separately for each time interval $[t_i, t_{i+1})$, $i \in \bar{\mathbb{N}}$. That is, when a new measurement $(x^{i+1}, \dot{x}^{i+1}, u^{i+1})$ is received, the estimation of $g(x)$ is updated, a new $\hat{g}^{i+1}$ is computed by Algorithm 1, and SafetyAdaptation is reset (j and ρ_ι are reset as in lines $1-3$). This is illustrated in the algorithm SafetyControl (Algorithm 3).

Algorithm 2 SafetyAdaptation($g^i, h, \bar{\varepsilon}, \underline{\varepsilon}$)

1: $\rho_\iota \leftarrow 1, \forall \iota \in \{1, \ldots, n+1\}$; $j \leftarrow 1$; $h_1 \leftarrow h_v$;
2: **while** True **do**
3:     **if** $\|\hat{g}^i(x)^\top \nabla_{x_2} h_j(x)\| \leq \underline{\varepsilon} \wedge \rho_j = 1$ **then**
4:         $x_c \leftarrow x$; $\rho_j \leftarrow 0$;
5:         Find h_{j+1} such that
(1) $\mathcal{C}(h_{j+1}) \subset \mathcal{C}(h_j)$
(2) $h_{j+1}(x_c) > 0$
(3) $\|\hat{g}^i(x_c)^\top \nabla_{x_2} h_{j+1}(x_c)\| \geq \gamma_{j+1}$;
6:         $j \leftarrow j+1$;
7:     **for** $\iota \in \{1, \ldots, j-1\}$ **do**
8:         **if** $\|\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)\| > \bar{\varepsilon} \wedge \rho_\iota = 0$ **then**
9:             $\rho_\iota \leftarrow 1$; $j \leftarrow i$;
10:            Break;
11:    Apply (17)

Algorithm 2 imposes an extra, state-dependent switching to the closed-loop system, which can be written as

$$\dot{x} = f(x) + g(x)u_{\rm n} - \kappa_v \sigma_{\mu_v}(h_v(x)) g(x) u_j(x,t), \quad (18)$$

where $u_j := \beta_{j,\rm d}(x) \frac{\hat{g}^i(x)^\top \nabla_{x_2} h_j(x)}{\|\hat{g}^i(x)^\top \nabla_{x_2} h_j(x)\|^2}$, for some $i \in \bar{\mathbb{N}}$, $j \in \mathbb{N}$. The switching regions are not pre-defined, but detected online based on the trajectory of the system (line 4 of Algorithm 2). Moreover, by choosing $\gamma_j > \underline{\varepsilon}$, for all $j \geq 2$, we guarantee that the switching does not happen continuously, and hence the solution of (18) is well-defined in $[t_0, t_{\max})$ for some $t_{\max} > t_0$, satisfying $x(t) \in \text{Int}(\bar{\mathcal{C}}) \cap \text{Int}(\mathcal{C}_v)$, for all $t \in [t_0, t_{\max})$ (similar to the proof of Lemma 2).

Theorem 4 guarantees the safety of the system by using Algorithm 3, by showing that, for each $i \in \bar{\mathbb{N}}$, the iterator j in Algorithm 2, indicating the number of h_j computed, reaches at most $2n+1$. We first define the sets $\mathcal{S}_j^i := \{x \in \bar{\mathcal{C}} \cap \text{Cl}(\mathcal{C}_{v,\mu_v}) : \hat{g}^i(x)^\top \nabla_{x_2} h_j(x) = 0\}$ $\mathfrak{S}_j^i := \bigcap_{q=1}^j \mathcal{S}_q^i$, for all $i \in \bar{\mathbb{N}}$, $j \in \{1, \ldots, 2n\}$, for which we impose the following assumption.

Algorithm 3 SafetyControl($\bar{\varepsilon}, \underline{\varepsilon}$)

$i \leftarrow 1$;
$\hat{g}^i \leftarrow$ rand(n, m);
while True **do**
  while NoNewMeasurement **do**
    SafetyAdaptation$(\hat{g}^i, h, \bar{\varepsilon}, \underline{\varepsilon})$
  $i \leftarrow i + 1$;
  $\hat{g}^i \leftarrow$ Approximate$(\dot{x}(t_i), x(t_i), u(t_i))$;

Assumption 5 *Let $i \in \bar{\mathbb{N}}$. The sets $\mathcal{S}_j^i$, $\mathfrak{S}_j^i$ are manifolds satisfying* $\dim(\mathcal{S}_j^i) \leq 2n-1$, $\forall j \in \{1, \ldots, 2n\}$, *and* $\text{codim}(\mathfrak{S}_j^i \cap \mathcal{S}_{j+1}^i) \geq \text{codim}(\mathfrak{S}_j^i) + \text{codim}(\mathcal{S}_{j+1}^i)$, $\forall j \in \{1, \ldots, 2n-1\}$.

Assumption 5 first states that $\mathcal{S}_j^i$ are lower-dimension manifolds (e.g., lines on the plane or planes in 3D space), which is common for curves like $\hat{g}^i(x)^\top \nabla_{x_2} h_j(x) = 0$. The dimension condition is essentially a mild transversality condition on the manifolds $\mathcal{S}_j^i$ [43], which implies that $\dim(\mathfrak{S}_j^i \cap \mathcal{S}_{j+1}^i) \leq \dim(\mathfrak{S}_j^i) + \dim(\mathcal{S}_{j+1}^i) - 2n$. We are now ready to state the main result of this section.

Theorem 4 *Let a system evolve according to the dynamics* (2) *and control law* (17). *Let also a set $\mathcal{C}$ satisfying $x_1(t_0) \in \text{Int}(\mathcal{C})$ for a positive $t_0 \geq 0$. Let T_μ and i_t as defined in Theorem 2.*

Assume there exists $\mu'_v \in (0, \mu_v)$ such that the following holds: for each $t \in T_{\mu'_v}$, there exist $r := r(t, \mu'_v) > 0$, $\varepsilon < 1$, for which the following conditions hold:

$$\mathcal{B}_r(x(t)) \subset \mathcal{C}_v \quad (19a)$$
$$\|\widetilde{g}^{i_t}(y)\| \|\nabla_{x_2} h_j(y)\| < \varepsilon \sigma_{\mu_v}(\mu'_v), \quad (19b)$$

for all $y \in \mathcal{B}_r(x(t))$, where j is the iterator variable of Algorithm 2, signifying $\rho_j = 1$, and $\rho_\iota = 0$, if $\iota \neq j$. Let Ass. 1, 3, and 5 hold. Then, under sufficiently large γ_{2n+1} and sufficiently small $\bar{\varepsilon}$, it holds that $x_1(t) \in \text{Int}(\mathcal{C})$, and all closed-loop signals are bounded, for all $t \geq t_0$.

We now briefly elaborate on the practical implications and complexity of Algorithm 2. Note that almost all operations the algorithm executes are standard multiplications and conditionals that can be evaluated almost instantaneously. The main computational bottleneck consists of that of the derivation of the new barrier functions h_{j+1} in line 6. We acknowledge that an efficient computation of h_{j+1} such that it allows the online execution of the algorithm is a challenging problem. One can employ standard optimization techniques to derive a conservative solution. An example consists of maximizing the distance $\|\hat{g}^i(x_c)^\top \nabla_{x_2} h_{j+1}(x_c)\| - \gamma_{n+1}$ (third property in line 6) subject to the constraints $\mathcal{C}(h_{j+1}) \subset \mathcal{C}(h_j)$ and $h_{j+1}(x_c) > 0$ (first two properties in line 6), or formulating a convex, more conservative version that favors computational efficiency. In the simulation results of Section 6, we employ a nonlinear optimization procedure that yields low enough computation time to allow for the online execution of the algorithm.

The efficient computation of h_{j+1} is even more important because of the fact that Algorithm 2 is reset at every t_i, $i \in \bar{\mathbb{N}}$. Therefore, small $\Delta t_i = t_{i+1} - t_i$, required for the accurate approximation of $g(x)$ (see Lemma 3), might hinder the successful computation of h_{j+1} and, consequently, the execution of Algorithm 2. A possible, practical solution would be to maintain the execution of Algorithm 2 until a significant change on $\hat{g}^i(x)$ occurs, i.e., until $\|\hat{g}^i(x) - \hat{g}^{i-1}(x)\|$ becomes large enough, instead of resetting it at every time instant t_i, $i \in \bar{\mathbb{N}}$.

6 Simulation Results

We validate the proposed algorithm with a simulation example. More specifically, we consider an underactuated unmanned aerial vehicle (UAV) with state variables $x = [x_1, \ldots, x_6] = [p_x, p_y, \phi, v_x, v_y, \omega]^\top$ evolving subject to the dynamics $\dot{p}_x = v_x$, $\dot{p}_y = v_y$, $\dot{\phi} = \omega$, and

$$m\dot{v}_x = -C_D^v v_x - u_1 \sin(\phi) - u_2 \sin(\phi)$$
$$m\dot{v}_y = -(mg + C_D^v v_y) + u_1 \cos(\phi) + u_2 \cos(\phi)$$
$$2I\dot{\omega} = -C_D^\phi \omega - l u_1 + l u_2,$$

where $m = 1.25$, $I = 0.03$ are the quadrotor's mass and moment of inertia, respectively, $g = 9.81$ is the gravity constant, $l = 0.5$ is the arm length, and $C_D^v = 0.25$, $C_D^\phi = 0.02255$ are aerodynamic constants. We consider that the UAV aims to track the helicoidal trajectory $p_{x_r} := \frac{1}{2}\sin(\frac{3}{2}t)$, $p_{y_r} := \frac{1}{2}\sin(\frac{3}{4}t)$ via an appropriately designed nominal control input $u_{\rm n}$. We wish to bound the position (p_x, p_y) of the UAV through the sphere $h(x) =$

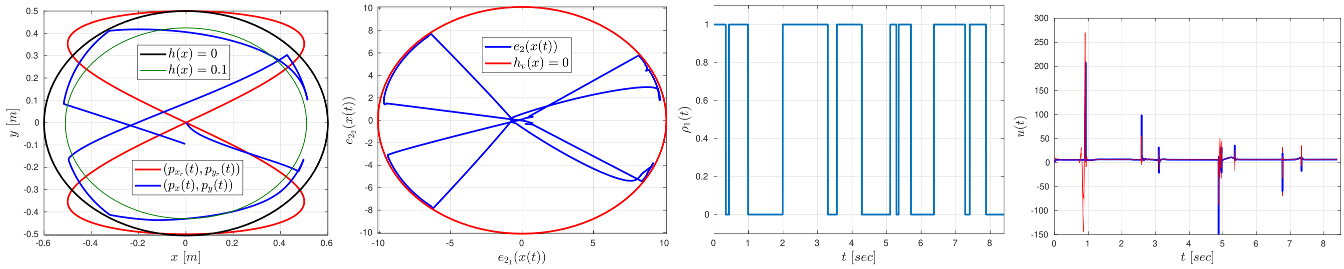

Fig. 3. From left to right: The reference trajectory $(p_{x_r}(t), p_{y_r}(t))$ (red), and the system trajectory $(p_x(t), p_y(t))$ (blue), along with the boundaries $h(x) = 0$ (black) and $h(x) = 0.1$ (green). The velocity error signal $e_2(x(t))$ (blue) and the boundary $h_v(x) = 0$ (red). The evolution of $\rho_1(t) \in \{0, 1\}$. The evolution of the control inputs $u_1(t)$ (red) and $u_2(t)$ (blue).

$0.36 - \|[p_x, p_y]^\top\|^2$. We use the local safety controller (17), with $\beta = \frac{1}{h}$, $\mu_x = 0.2$, $\kappa_x = 0.1$, $h_v(x) = 100 - e_2^2$, $\beta_v := \frac{1}{h_v}$, $\mu_v = 0.1$, $\kappa_v = 0.1$, while setting $\underline{\varepsilon} = 0.05$, $\bar{\varepsilon} = 5$ in Algorithm 2. The data measurement and hence the execution of Algorithm 1 occurs every 0.1 seconds. For the case when $\|\hat{g}^i(x_c)^\top \nabla x_2 h_j(x_c)\| \leq \underline{\varepsilon}$ for some x_c, we use an optimization solver that aims to find an ellipsoidal $h_{j+1}(x)$ such that $\mathcal{C}_{j+1} \subset \mathcal{C}_j$ and maximize the value $\|\hat{g}^i(x_c)^\top \nabla x_2 h_{j+1}(x_c)\|$. The computation time used by the solver does not exceed 0.01 seconds, allowing the online execution of Algorithm 2.

The simulation results from the initial condition $[0, 0.2, 0, -0.3, 0, 0]^\top$ are illustrated in Fig. 3 for $t \in [0, 8.5]$ seconds. The figure depicts the reference (red) and the system trajectory (blue) along with the boundaries of the barrier $h(x) = 0$ (black) and local barrier function $h(x) = \mu = 0.1$ (green). One can verify that the system position is successfully confined in the set $\text{Int}(\mathcal{C})$ defined by $h(x) > 0$, verifying thus the theoretical findings. The figure further depicts the evolution of the error $e_2(x(t)) = [e_{2_1}(x(t)), e_{2_2}(x(t))]$, which is successfully confined in the sphere imposed by $h_2(x)$. Finally, the figure depicts the evolution of $\rho_1(t)$ and the required control input. It is concluded that $\|g^i(x)^\top \nabla_{x_2} h_v(x)\|$ falls below $\underline{\varepsilon}$ several times and a new function $h_2(x)$ is found, as per Algorithm 2. While h_2 is activated, however, $\|g^i(x)^\top \nabla_{x_2} h_2(x)\|$ is always above $\underline{\varepsilon}$, not requiring thus a new $h_3(x)$.

7 Conclusion and Future Work

We consider the safety problem for a class of 2nd-order nonlinear unknown systems. We propose a two-layered control solution, integrating approximation of dynamics from limited data with closed-form nonlinear control laws using reciprocal barriers. Future efforts will be devoted towards relaxing the assumption on the approximation error $\tilde{g}^i(x)$, establishing persistence of excitation conditions, and extending the proposed framework to stabilization/tracking with input constraints.

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A Proofs

Proof of Lemma 1: By the definition of G^i_{kl}, given by the output $\mathscr{E}_i$ of Algorithm 1 and Lemma 2 of [4], we have for any $x \in \bar{\mathcal{A}}$ that

$$\underline{\mathsf{G}}^i_{k\ell}(x) = \inf \bigcap_{(x^j,\cdot,C^j_{\mathcal{G}_{k\ell}}) \in \mathscr{E}_i} \left\{ C^j_{\mathcal{G}_{k\ell}} + \bar{g}_{k\ell} ||x - x^j|| [-1, 1] \right\},$$

$$\bar{\mathsf{G}}^i_{k\ell}(x) = \sup \bigcap_{(x^j,\cdot,C^j_{\mathcal{G}_{k\ell}}) \in \mathscr{E}_i} \left\{ C^j_{\mathcal{G}_{k\ell}} + \bar{g}_{k\ell} ||x - x^j|| [-1, 1], \right\}$$

for all $k \in \{1,\ldots,n\}$ and $\ell \in \{1,\ldots,m\}$. Thus, one can observe that $\underline{\mathcal{G}}^i_{k\ell}(x)$ and $\bar{\mathcal{G}}^i_{k\ell}(x)$ can be also written as

$$\underline{\mathsf{G}}^i_{k\ell}(x) = \inf_{(x^j,\cdot,C^j_{\mathcal{G}_{k\ell}})\in\mathscr{E}_i} \left\{m_\mathrm{u}(x^j) + \bar{g}_{k\ell}\|x - x^j\|\right\},$$
$$\bar{\mathsf{G}}^i_{k\ell}(x) = \sup_{(x^j,\cdot,C^j_{\mathcal{G}_{k\ell}})\in\mathscr{E}_i} \left\{m_\mathrm{l}(x^j) - \bar{g}_{k\ell}\|x - x^j\|\right\},$$

where $m_\mathrm{l}(x^j) := \inf C^j_{\mathcal{G}_{k\ell}}$ and $m_\mathrm{u}(x^j) := \sup C^j_{\mathcal{G}_{k\ell}}$. Using the inequality $|\|\mathsf{x}\| - \|\mathsf{y}\|| \le \|\mathsf{x}-\mathsf{y}\|$ for any two vectors $\mathsf{x},\mathsf{y} \in \mathbb{R}^n$, we conclude that the functions $h^j : \mathsf{x} \mapsto m_\mathrm{u}(x^j) + \bar{g}_{k\ell}\|\mathsf{x} - x^j\|$ and $l^j : \mathsf{x} \mapsto m_\mathrm{l}(x^j) - \bar{g}_{k\ell}\|\mathsf{x} - x^j\|$ are Lipschitz continuous in $\bar{\mathcal{A}}$ with $\bar{g}_{k\ell}$ as the Lipschitz constant in $\bar{\mathcal{A}}$. Similarly, and by using $\max\{l,h\} = 0.5(l+h+|l-h|)$, and $\min\{l,h\} = 0.5(l+h-|l-h|)$, we can deduce by induction that $\mathsf{x} \mapsto \sup_{j\in\{1,\ldots,i\}} l_j(\mathsf{x})$ and $\mathsf{x} \mapsto \inf_{j\in\{1,\ldots,i\}} h_j(\mathsf{x})$ are also Lipschitz continuous owing to the Lipschitz continuity of l^j and h^j for all $j \in \{1,\ldots,n\}$. Thus, one obtains that $x \mapsto \bar{\mathsf{G}}^i_{k\ell}(x)$ and $x \mapsto \underline{\mathsf{G}}^i_{k\ell}(x)$ are also Lipschitz continuous, from which we conclude the Lipschitz continuity of $\hat{g}_{k\ell}$.

Proof of Lemma 2 The closed-loop system $\dot{x} = f(x) + g(x)u(x,t)$ is piecewise continuous in $t \ge t_0$, for each fixed $x \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$, and, in view of Lemma 1, locally Lipschitz in $x \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$ for each fixed $t \ge t_0$. Hence, since $\mathcal{C}_v$ is designed such that $x(t_0) \in \mathrm{Int}(\mathcal{C}_v)$, we conclude from [44, Theorem 2.1.3] the existence of a unique, maximal, and absolutely continuous solution $x(t)$, satisfying $x(t) \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$, for all $t \in [t_0, t_{\max})$, for a positive constant $t_{\max} > t_0$.

Proof of Theorem 2 According to Lemma 2, it holds that $x(t) \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$, for all $t \in [t_0, t_{\max})$ for a $t_{\max} > t_0$. Assume now that $\lim_{t\to t_{\max}} h(x_1(t)) = 0$, i.e., the system converges to the boundary of $\mathcal{C}$ as $t \to t_{\max}$, implying $\lim_{t\to t_{\max}} \beta(h(x_1(t))) = \infty$. Let $t'_x \in [t_0, t_{\max})$ and $\mu'_x \in (0, \mu_x)$ such that $x_1(t) \in \mathrm{Int}(\mathcal{C}_{\mu'_x}) \subset \mathrm{Int}(\mathcal{C}_{\mu_x})$ for all $t \in [t'_x, t_{\max})$, and $x_1(t) \in \widetilde{\mathcal{C}}_x := \{x_1 \in \mathbb{R}^n : h(x_1) \ge \min_{t\in[t_0,t']}\{h(x_1(t))\} > 0\}$, for all $t \in [t_0, t'_x]$ [2]. Hence, it holds $0 < h(x_1(t)) \le \mu'_x < \mu_x$ and $\sigma_{\mu_x}(h(x_1(t))) > \sigma_{\mu_x}(\mu'_x) > 0$, for all $t \in [t'_x, t_{\max})$. In view of (9) and (10), $\dot{\beta}$ becomes

$$\dot{\beta} = \beta_\mathrm{d} \nabla h(x_1)^\top e_2 - \kappa_x \sigma_{\mu_x}(h(x_1))\beta_\mathrm{d}^2 \|\nabla h(x_1)\|^2.$$

In view of the definition of h_v and since $x(t) \in \mathrm{Int}(\mathcal{C}_v)$ for $t \in [t_0, t_{\max})$, we conclude that $\|e_2(x(t)))\| < \bar{B}_2$, for $t \in [t_0, t_{\max})$. Moreover, since $\mu'_x < \mu_x < \nu_h$, Assumption 3 suggests that $\|\nabla h(x_1(t))\| \ge \varepsilon_h$ for all $t \in [t'_x, t_{\max})$. Therefore, $\dot{\beta}$ becomes

$$\dot{\beta} \le -\kappa_x \sigma_{\mu_x}(\mu'_x)\varepsilon_h^2 |\beta_\mathrm{d}| \left(|\beta_\mathrm{d}| - \frac{\bar{B}_2 \bar{h}_x}{\kappa_v \sigma_{\mu_x}(\mu'_x)\varepsilon_h^2}\right) \quad \text{(A.1)}$$

[2] Note that such t'_x, μ'_x exist since $\lim_{t\to t_{\max}} h(x_1(t)) = 0$.

for all $t \in [t'_x, t_{\max})$, where $\bar{h}_x := \sup_{x_1\in\mathcal{C}_{\mu'_x}} \|\nabla h(x_1)\|$ is a finite constant, since $h(x_1)$ is continuously differentiable. Therefore, $\dot{\beta} < 0$ when $|\beta_\mathrm{d}| > \frac{\bar{B}_2 \bar{h}_x}{\kappa_x \sigma_{\mu_x}(\mu'_x)\varepsilon_h^2}$.

We claim now that (A.1) implies the boundedness of β. Since we have assumed that $\lim_{t\to t_{\max}} \beta(h(x_1(t))) = \infty$, (7) and (8) imply that $\lim_{t\to t_{\max}} |\beta_\mathrm{d}(t)| = \infty$. Hence, for every positive constant $\gamma > 0$, there exists a time instant $t_\gamma \in [t'_x, t_{\max})$ such that $|\beta_\mathrm{d}(t)| > \gamma$ for all $t > t_\gamma$. Consequently, we conclude from (A.1) that there exists a time instant $t' \in [t'_x, t_{\max})$ such that $\beta(h(x_1(t))) < 0$ for all $t > t'$, which leads to a contradiction. We conclude, therefore, that there exists a constant $\bar{\beta}$ such that $\beta(h(x_1(t)) \le \bar{\beta}$, for all $t \in [t_0, t_{\max})$, implying $h(x_1(t)) \ge \underline{h} := \alpha_1^{-1}\left(\frac{1}{\bar{\beta}}\right)$, for all $t \in [t_0, t_{\max})$, which dictates the boundedness of x_1 in a compact set $x_1(t) \in \widetilde{\mathcal{C}} \subset \mathrm{Int}(\mathcal{C})$, for all $t \in [t_0, t_{\max})$. Moreover, (9) also suggests the boundedness of $x_{2,\mathrm{r}}(x_1(t))$, which, via the boundedness of e_2 by $\bar{B}_2$, implies the boundedness of $x_2(t)$, for all $t \in [t_0, t_{\max})$. By differentiating (9) and using the boundedness of x_1, $x_{2,\mathrm{r}}$, and (8), (10), we also conclude the boundedness of $\dot{x}_{2,\mathrm{r}}(x_1(t)))$, for all $t \in [t_0, t_{\max})$.

We proceed next to prove the boundedness of β_v. Following the same line of proof, assume that $\lim_{t\to t_{\max}} h_v(x(t)) = 0$, i.e., the system converges to the boundary of $\mathcal{C}_v$ as $t \to t_{\max}$, implying $\lim_{t\to t_{\max}} \beta_v(h_v(x(t))) = \infty$. Given the constant $\mu'_v \in (0, \mu_v)$, let any $t'_v \in [t_0, t_{\max})$ such that $x(t) \in \mathrm{Int}(\mathcal{C}_{v,\mu'_v}) \subset \mathrm{Int}(\mathcal{C}_{v,\mu_v})$ for all $t \in [t'_v, t_{\max})$, and $x(t) \in \widetilde{\mathcal{C}}_v := \{x \in \mathbb{R}^{2n} : h_v(x) \ge \min_{t\in[t_0,t']}\{h_v(x(t))\} > 0\}$, for all $t \in [t_0, t'_v]$. Hence, it holds $0 < h_v(x(t)) \le \mu'_v < \mu_v$ and $\sigma_{\mu_v}(h_v(x(t))) > \sigma_{\mu_v}(\mu'_v) > 0$, for all $t \in [t'_v, t_{\max})$. Since $\sigma_{\mu_v}(h_v) \le 1$ and $h_v(\cdot)$ is a function of e_2, $\dot{\beta}_v$ becomes

$$\dot{\beta}_v \le \beta_{v,\mathrm{d}} f_\mathrm{n}(x) - \kappa_v \sigma_{\mu_v}(h_v)\beta_{v,\mathrm{d}}^2 + \kappa_v \beta_{v,\mathrm{d}}^2 \frac{\widetilde{g}^i(x)^\top \nabla_{x_2} h_v(x)}{\|\hat{g}^i(x)^\top \nabla_{x_2} h_v(x)\|}$$

for $t \in [t'_v, t_{\max})$, where $f_\mathrm{n}(x) := \nabla_{x_1} h_v(x)^\top x_2 + \nabla_{x_2} h_v(x)(f(x) + g(x)u_\mathrm{n}(x))$ and we used $g(x) = \hat{g}^i(x) - \widetilde{g}^i(x)$. Since f, g, u_n are continuous, $\dot{x}_{2,\mathrm{r}}$ has been proven bounded, and $x(t) \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$ for $t \in [t_0, t_{\max})$, there exists a constant $\bar{f}_\mathrm{n}$, independent of $t_{\max}$, satisfying $|f_\mathrm{n}(x(t))| \le \bar{f}_\mathrm{n}$, for $t \in [t_0, t_{\max})$. Further, since $x(t) \in \mathrm{Int}(\mathcal{C}_{v,\mu'_v})$ for $t \in [t'_v, t_{\max})$, it holds that $[t'_v, t_{\max}) \subset T_{\mu'_v}$. Hence, in view of (13), we obtain

$$\dot{\beta} \le -\kappa_v \sigma_{\mu_v}(\mu'_v)(1-\varepsilon)|\beta_{v,\mathrm{d}}| \left(|\beta_{v,\mathrm{d}}| - \frac{\bar{f}_\mathrm{n}}{\kappa_v \sigma_{\mu_v}(\mu'_v)(1-\varepsilon)}\right)$$

for all $t \in [t'_v, t_{\max})$, where we define $\varepsilon := \frac{\varepsilon_2}{\varepsilon_1} < 1$. Therefore, $\dot{\beta}_v < 0$ when $|\beta_{v,\mathrm{d}}| > \frac{\bar{f}_\mathrm{n}}{\kappa_v \sigma_{\mu_v}(\mu'_v)(1-\varepsilon)}$. By invoking similar arguments as in the case of (A.1) and β, we conclude that $x(t) \in \widetilde{\mathcal{C}} \subset \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$. By also using $x(t) \in \widetilde{\mathcal{C}}_v$, for all $t \in [t_0, t'_v]$ and the compactness of $\widetilde{\mathcal{C}}_v$, we conclude the boundedness of $x(t)$ and $\beta_v(h_v(x(t)))$, for all $t \in [t_0, t_{\max})$. From [44, Th. 2.1.4], we conclude that $t_{\max} = \infty$, and the boundedness of $x(t)$, $\beta(h(x_1(t)))$, $u(x(t), t)$ for all $t \in [t_0, \infty)$.

Proof of Theorem 3 The proof follows similar steps as in the proof of Theorem 2 and only a sketch is given. Firstly, we establish a unique, continuously differentiable, and maximal solution $x : [t_0, t_{\max}) \to \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathrm{Int}(\mathcal{C}_v)$, for some $t_{\max} > t_0$. By differentiating β, we obtain (A.1), which guarantees the boundedness of β as $\beta(h(x_1(t)) \leq \bar{\beta}$ and the boundedness of x_1, x_2, $x_{2,\mathrm{r}}$, and $\dot{x}_{2,\mathrm{r}}$ for all $t \in [t_0, t_{\max})$.

Proceeding similarly as in the proof of Theorem 2, we assume that $\lim_{t \to t_{\max}} h_v(x(t)) = 0$ and consider a constant $t'_v \in [t_0, t_{\max})$ such that $x(t) \in \mathcal{C}_{v,\mu'_v}$ for all $t \in [t'_v, t_{\max})$, implying $\sigma_{\mu_v}(h_v(x(t))) > \sigma_{\mu_v}(\mu'_v) > 0$, for all $t \in [t'_v, t_{\max})$. Next, we define the continuous function $f_{\mathrm{n}}(x) := \nabla_{x_1} h_v(x)^\top x_2 + \nabla_{x_2} h_v(x)(f(x) + g(x)u_{\mathrm{n}}(x))$, which is bounded by a constant $\bar{f}_{\mathrm{n}}$, for $t \in [t'_v, t_{\max})$. Further, since $x(t) \in \mathcal{C}_{\mu'_v} \subset \mathcal{C}_{\mu_v}$ for $t \in [t'_v, t_{\max})$, it holds that $\|\nabla_{x_2} h_v(x)\| \geq \varepsilon_v > 0$. Finally, by using the identity $g(x) = \frac{1}{2}(g(x) + g(x)^\top) + \frac{1}{2}(g(x) - g(x)^\top)$, and employing the skew symmetry of $g(x) - g(x)^\top$ and the positive definiteness of $g(x) + g(x)^\top$, we obtain

$$\dot{\beta}_v \leq -\epsilon_v \kappa_v \underline{g} \sigma_{\mu_v}(\mu'_v) |\beta_{v,\mathrm{d}}| \left(|\beta_{v,\mathrm{d}}| - \frac{\bar{f}_{\mathrm{n}}}{\epsilon_v \kappa_v \underline{g} \sigma_{\mu_v}(\mu'_v)} \right),$$

where $\underline{g}$ is the minimum eigenvalue of $g(x)+g(x)^\top$, which is positive for all $t \in [t'_v, t_{\max})$. Therefore, we conclude that $\dot{\beta}_v > 0$ when $|\beta_{v,\mathrm{d}}| > \frac{\bar{f}_{\mathrm{n}}}{\varepsilon_v \kappa_v \underline{g} \sigma_{\mu_v}(\mu'_v)}$. By proceedings similarly to the proof of Theorem 2, the proof follows.

To prove Lemma 3, we need the following result from [42].

Theorem 5 ([42], Theorem 2) *Let the current state be x^i, the bounded admissible set of control values between time t_i and $t_{i+1} = t_i + \Delta t_i$ be $\mathcal{U}^i \in \mathbb{IR}^m$, with a time step size $\Delta t_i > 0$. Assume that $(\sqrt{n}\delta^i)\Delta t < 1$, where $\delta^i := \sqrt{\sum_{k=1}^n (\bar{f}_k + \sum_{\ell=1}^m \bar{g}_{k\ell} |\mathcal{U}_l^i|)^2}$, and $\bar{f}_k$, $\bar{g}_{k\ell}$ are the known locally Lipschitz constants (see Assumption 2). Then, the future state value $x^{i+1} = x(t_{i+1})$ satisfies*

$$x^{i+1} \in x^i + \boldsymbol{h}(x^i, \mathcal{U}^i)\Delta t + (\mathcal{J}^f + \mathcal{J}^g \mathcal{U}^i)\boldsymbol{h}(\mathcal{S}^i, \mathcal{U}^i)\frac{\Delta t_i^2}{2}$$

Note that, with a slight abuse of notation, $\boldsymbol{F}$ and $\boldsymbol{G}$ are re-defined in Theorem 5 to take both real vectors and interval quantities as arguments, as expressed by $\boldsymbol{h}(x^i, \mathcal{U}^i)$ and $\boldsymbol{h}(\mathcal{S}^i, \mathcal{U}^i)$. To achieve this, one can straightforwardly extend the $\|\cdot\|$ operator to the domain of intervals [4].

We are now ready to prove Lemma 3.

Proof of Lemma 3 By definition of $\hat{g}_{k\ell}(x) \in \boldsymbol{G}_{k\ell}(x)$, we have

$$\|\hat{g}_{k\ell}(x^{i+1}) - g_{k\ell}(x^{i+1})\| \leq \mathrm{wd}(\boldsymbol{G}_{k\ell}(x^{i+1})) \quad \text{(A.2)}$$

since we know by construction that $g_{k\ell}(x^{i+1}) \in \boldsymbol{G}_{k\ell}(x^{i+1})$ for all $k \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, m\}$. As a consequence, by construction of $\boldsymbol{G}_{k\ell}$ in Theorem 1 and Lemma 2 of [4], we can deduce that

$$\mathrm{wd}(\boldsymbol{G}_{k\ell}(x^{i+1})) \leq \mathrm{wd}(C^i_{\mathcal{G}_{k\ell}} + \bar{g}_{k\ell}\|x^{i+1} - x^i\|)][-1, 1]). \quad \text{(A.3)}$$

In view of Theorem 5,

$$\|x^{i+1} - x^i\| \in \left\| \boldsymbol{h}(x^i, \mathcal{U}^i)\Delta t_i + (\mathcal{J}^f + \mathcal{J}^g \mathcal{U}^i)\boldsymbol{h}(\mathcal{S}^i, \mathcal{U}^i)\frac{\Delta t_i^2}{2} \right\|. \quad \text{(A.4)}$$

Then, using the definition of $\mathcal{S}^i$ from (14), we have that

$$\boldsymbol{F}_k(\mathcal{S}^i) \subseteq C^i_{\mathcal{F}_k} + \bar{f}_k \frac{\Delta t_i \|\boldsymbol{h}(x^i, \mathcal{U}^i)\|_\infty}{1 - \sqrt{n}\Delta t_i \delta^i}[-\sqrt{n}, \sqrt{n}],$$

$$\boldsymbol{G}_{k,l}(\mathcal{S}^i) \subseteq C^i_{\mathcal{G}_{k\ell}} + \bar{g}_{k\ell} \frac{\Delta t_i \|\boldsymbol{h}(x^i, \mathcal{U}^i)\|_\infty}{1 - \sqrt{n}\Delta t_i \delta^i}[-\sqrt{n}, \sqrt{n}].$$

Hence, $\boldsymbol{h}(\mathcal{S}^i, \mathcal{U}^i) \subseteq \boldsymbol{h}(x^i, \mathcal{U}^i) + \frac{\Delta t_i \|\boldsymbol{h}(x^i, \mathcal{U}^i)\|_\infty}{1 - \sqrt{n}\Delta t_i \delta^i}\mathcal{H}^i$, which, after merging with (A.4), and plugging the result into (A.3) and then (A.2), enables to obtain (15).

We next proceed to prove Theorem 4. Given the sets $\mathcal{S}^i_j$, $\mathfrak{S}^i_j$, we first define the inflated sets and their intersections $\widetilde{\mathcal{S}}^i_j(\bar{\varepsilon}) := \{x \in \bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v}) : \|\hat{g}^i(x)^\top \nabla_{x_2} h_j(x)\| \leq \bar{\varepsilon}\}$ and $\widetilde{\mathfrak{S}}^i_j(\bar{\varepsilon}) := \bigcap_{q=1}^j \widetilde{\mathcal{S}}^i_q(\bar{\varepsilon})$, for $j \in \{1, \ldots, 2n\}$. We note that, based on the defined maximal solution, $x(t)$ evolves in $\mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v} \subset \bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$ for $t \in [t_0, t_{\max})$. Nevertheless, we employ the closed set $\bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$ in the aforementioned definitions to ease the following technical presentation and avoid notational jargon.

Similar to (12), (17) implies that the safety controller is activated close to the boundary of $\mathcal{C}_v$, i.e., in $\mathcal{C}_{v,\mu_v}$. In what follows, we focus on the solution parts that belong to $\mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v}$. In particular, let an $i \in \bar{\mathbb{N}}$ and τ_1, τ_2, such that $\tau_1 \geq t_i$, $\tau_2 \in (\tau_1, t_{i+1})$ and $x(t) \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v}$ for all $t \in [\tau_1, \tau_2)$. Note that τ_2 can either be smaller than $t_{\max}$, implying that the system navigates, at $t = \tau_2$, to $\mathrm{Int}(\bar{\mathcal{C}}) \cap (\mathcal{C}_v \backslash \mathcal{C}_{v,\mu_v})$, or $\tau_2 = t_{\max}$, implying that the system converges to the system boundary as $t \to t_{\max}$. That is, we examine the part of the solution that belongs to the set $\mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v}$ between two consecutive updates.

Let the solution restriction $\widetilde{x}(t) := x(t)$ for $t \in [\tau_1, \tau_2)$. Then it holds that $\widetilde{x}(t) \in \mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v}$, and $\widetilde{x}(t)$ can be decomposed based on the iterator j of Algorithm 2 as

$$\widetilde{x}(t) = \begin{cases} y_{1,I_1}(t), \quad t \in T_{1,1} = [\mathfrak{l}_{1,I_1}, \mathfrak{r}_{1,I_1}), \\ y_{1,I_1+1}(t), \quad t \in T_{1,I_1+1} = [\mathfrak{l}_{1,I_1+1}, \mathfrak{r}_{1,I_1+1}), \\ \dots \\ y_{1,F_1}(t), \quad t \in T_{1,F_1} = [\mathfrak{l}_{1,F_1}, \mathfrak{r}_{1,F_1}), \\ y_{2,I_2}(t), \quad t \in T_{2,I_2} = [\mathfrak{l}_{2,I_2}, \mathfrak{r}_{2,I_2}), \\ y_{2,I_2+1}(t), \quad t \in T_{2,I_2+1} = [\mathfrak{l}_{2,I_2+1}, \mathfrak{r}_{2,I_2+1}), \\ \dots \\ y_{2,F_2}(t), \quad t \in T_{2,F_2} = [\mathfrak{l}_{2,F_2}, \mathfrak{r}_{2,F_2}), \\ y_{3,I_3}(t), \quad t \in T_{3,I_3} = [\mathfrak{l}_{3,I_3}, \mathfrak{r}_{3,I_3}), \\ y_{3,I_3+1}(t), \quad t \in T_{3,I_3+1} = [\mathfrak{l}_{3,I_3+1}, \mathfrak{r}_{3,I_3+1}), \\ \dots \end{cases} \tag{A.5}$$

with $\mathfrak{l}_{1,I_1} = \tau_1$, $I_1 = 1$. The signal $y_{\ell,j}(t)$ stands for the solution when $h_j(x)$ is active after the $(\ell-1)$th reset of the SafetyAdaptation algorithm in line 15 [3]. Note also that ℓ is finite due to the hysteresis mechanism and the fact that $\gamma_j > \underline{\varepsilon}$; We denote by $\bar{\ell}$ its maximum value. The indices F_ℓ, I_ℓ are the last and first values of j (defining $h_j(x)$) at the $(\ell-1)$th reset (with $I_1 = 1$). The respective time intervals are defined as

$$\begin{aligned} &\mathfrak{r}_{\ell,j} = \mathfrak{l}_{\ell,j+1} := \inf\{t \in T_{\ell,j} : \\ &\quad \|\hat{g}^i(x(t))^\top \nabla_{x_2} h_j(x(t))\| \le \underline{\varepsilon}\}, \forall j \in \{I_\ell, \dots, F_\ell - 1\} \\ &\mathfrak{r}_{\ell,F_\ell} = \mathfrak{l}_{\ell+1,I_{\ell+1}} := \inf\{t \in T_{\ell,F_\ell} : \\ &\quad \exists \iota \in \{1, \dots, F_\ell\} \text{ s.t. } \|\hat{g}^i(x(t))^\top \nabla_{x_2} h_\iota(x(t))\| > \bar{\varepsilon}\}, \end{aligned}$$

with $\mathfrak{l}_{1,I_1} = \tau_i$, $I_1 = 1$. Note that ι from the definition of $\mathfrak{r}_{\ell,F_\ell}$ is equal to $I_{\ell+1}$ (see line 11 of Algorithm 2).

It holds that $T_{\ell,j} \subset [\tau_1, \tau_2)$ for all $\ell \in \{1, \dots, \bar{\ell}\}$, $j \in [I_\ell, \dots, F_\ell]$. Moreover, it holds that $y_{\ell,j}(t) \in \widetilde{\mathfrak{S}}^i_{j-1}(\bar{\varepsilon})$, $\forall t \in T_{\ell,j}$, for all $\ell \in \{1, \dots, \bar{\ell}\}$, $j \in \{I_\ell, \dots, F_\ell\}$, where $\widetilde{\mathfrak{S}}^i_0(\bar{\varepsilon}) := \bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v}) \backslash \bigcup_{j \in \{1,\dots,2n\}} \widetilde{\mathcal{S}}^i_j(\bar{\varepsilon})$.

In view of Assumption 5 and since $\mathrm{Int}(\bar{\mathcal{C}}) \cap \mathcal{C}_{v,\mu_v}$ is bounded, $\widetilde{\mathfrak{S}}^i_j(\bar{\varepsilon})$ are constituted by the union of a finite number (at least 1) of connected components, where $\|\hat{g}^i(x)^\top \nabla_{x_2} h_\iota(x)\| \le \bar{\varepsilon}$ holds for all $\iota \in \{1, \dots, j\}$, i.e.,

$$\widetilde{\mathfrak{S}}^i_j(\bar{\varepsilon}) := \bigcup_{q \in \mathcal{L}^i_j} \widetilde{\mathcal{K}}^{i,j}_q(\bar{\varepsilon})$$

for a finite index set $\mathcal{L}^i_j \subset \mathbb{N}$, $j \in \{1, \dots, 2n\}$. Since $\widetilde{\mathcal{K}}^{i,j}_q$ are closed, there exists a λ^* such that $\widetilde{\mathcal{K}}^{i,j}_{q_1}(\lambda^*) \cap \widetilde{\mathcal{K}}^{i,j}_{q_2}(\lambda^*) = \emptyset$, for $q_1, q_2 \in \mathcal{L}^i_j$, $q_1 \neq q_2$, $j \in \{1, \dots, 2n\}$.

[3] The 0th reset corresponds to the initial time period when no reset has occurred.

We show now that, by choosing a small enough $\bar{\varepsilon}$, each trajectory part $y_{\ell,j}(t)$ lies only in one of the $\widetilde{\mathcal{K}}^{\ell,j}_q$, for $j \in \{1, \dots, 2n\}$.

Proposition 1 *Let $\ell \in \{1, \dots, \bar{\ell}\}$, $j \in \{I_\ell, \dots, F_\ell\}$, $I_\ell \ge 1$ and assume that $F_\ell \le 2n + 1$. Then the choice $\bar{\varepsilon} < \frac{\lambda^*}{\sqrt{2n}}$ guarantees that there exists a $q^* \in \mathcal{L}^i_{j-1}$ such that $y_{\ell,j}(t) \in \widetilde{\mathcal{K}}^{i,j-1}_{q^*}(\lambda^*)$, implying $y_{\ell,j}(t) \notin \widetilde{\mathcal{K}}^{i,j-1}_q(\lambda^*)$, $\forall q \in \mathcal{L}^i_{j-1} \backslash \{q^*\}$.*

PROOF. Note first that $\sum_{\iota=1}^{j-1} \|\hat{g}^i(y_{\ell,j}(t))^\top \nabla_{x_2} h_\iota(y_{\ell,j}(t))\|^2 \le (j-1)\bar{\varepsilon}^2$, since the latter forms the circumscribed hyperellipsoid of the rectangular cuboid $\widetilde{\mathfrak{S}}^i_{j-1}(\bar{\varepsilon}) = \bigcap_{q=1}^{j-1} \widetilde{\mathcal{S}}^i_q(\bar{\varepsilon})$. Since $j \le 2n + 1$, by choosing $\bar{\varepsilon} \le \frac{\lambda^*}{\sqrt{2n}} \le \frac{\lambda}{\sqrt{j-1}}$, we guarantee that $\sum_{\iota=1}^{j-1} \|\hat{g}(y_{\ell,j}(t))^\top \nabla_{x_2} h_\iota(y_{\ell,j}(t))\|^2 \le (\lambda^*)^2$, which is the inscribed hyperellipsoid of the cuboid $\widetilde{\mathfrak{S}}^i_{j-1}(\lambda^*)$. Hence, $\|\hat{g}^i(y_{\ell,j}(t))^\top \nabla_{x_2} h_\iota(y_{\ell,j}(t))\| \le \lambda^*$, for all $\iota \in \{1, \dots, j-1\}$. Since $\widetilde{\mathcal{K}}^{i,j-1}_q(\lambda^*)$ are disjoint, $y_{\ell,j}(t)$ belongs to only one $\widetilde{\mathcal{K}}^{i,j-1}_{q^*}(\lambda^*)$, for some $q^* \in \mathcal{L}^i_{j-1}$, and $y_{\ell,j}(t) \notin \widetilde{\mathcal{K}}^{i,j-1}_q(\lambda^*)$, $\forall q \in \mathcal{L}^i_{j-1} \backslash \{q^*\}$.

By using Proposition 1 and Assumption 5, we prove next that by choosing a sufficiently large γ_{n+1}, we guarantee that the iterator variable j of Algorithm 2 is bounded.

Proposition 2 *There exist positive constants γ, ω such that, if $\bar{\varepsilon} < \omega$ and $\gamma_{2n+1} \ge \gamma$, there are no $t \ge t_0$ and $j \ge 2n + 1$ such that $\|g^i(x(t))^\top \nabla_{x_2} h_j(x(t))\| \le \underline{\varepsilon}$.*

PROOF. Let $j = 2n + 1$ in Algorithm 2, i.e.,

$$x(t) = y_{\ell,2n+1}(t) \in \widetilde{\mathfrak{S}}^i_{2n}(\bar{\varepsilon}) = \bigcap_{q=1}^{2n} \widetilde{\mathcal{S}}^i_q(\bar{\varepsilon}), \quad t \in T_{\ell,2n+1} \tag{A.6}$$

for some $\ell \in \{1, \dots, \bar{\ell}\}$. Assume that $\mathfrak{S}^i_{2n} = \emptyset$. Since $\mathcal{S}^i_q$ are closed, it can be concluded that there exists a positive constant ω such that $\widetilde{\mathfrak{S}}^i_{2n}(\omega) = \emptyset$. Hence, by choosing $\bar{\varepsilon} \le \omega$, we guarantee that $\widetilde{\mathfrak{S}}^i_{2n}(\bar{\varepsilon}) = \emptyset$, which contradicts (A.6). Hence, we conclude that $\mathfrak{S}^i_{2n} \neq \emptyset$, which, in view of Assumption 5, implies that $\dim(\mathfrak{S}^i_{2n}) = 0$. Therefore, the set $\mathfrak{S}^i_{2n}$ is a zero-dimensional manifold consisting of a finite set of points $\{p_1, p_2, \dots, p_m\}$ of $\mathbb{R}^{2n}$ for some $m \in \mathbb{N}$. The set $\widetilde{\mathfrak{S}}^i_{2n}(\bar{\varepsilon})$ is the intersection of $\bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$ with a union of closed rectangular hypercuboids around these points. In particular, based on the discussion prior to Prop. 1, $\widetilde{\mathfrak{S}}^i_{2n}(\bar{\varepsilon}) = \bigcup_{q \in \mathcal{L}^i_{2n}} \widetilde{\mathcal{K}}^{i,2n}_q(\bar{\varepsilon})$, where $\widetilde{\mathcal{K}}^{i,2n}_q(\bar{\varepsilon})$ are the intersections of these closed hypercuboids with $\bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$. According to Prop. 1, by choosing $\bar{\varepsilon}$ small enough, $\widetilde{K}^{i,2n}_q(\bar{\varepsilon})$ are disjoint, and hence $y_{\ell,2n+1}(t)$ evolves in

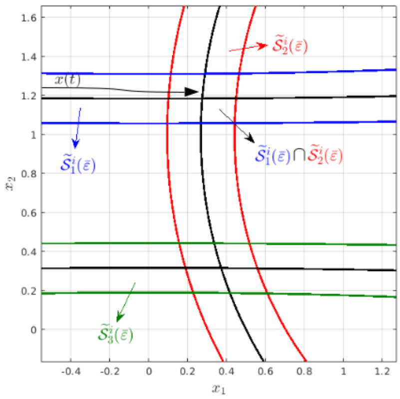

Fig. A.1. Illustration of Proposition 2. When $x(t)$ navigates to $\widetilde{\mathfrak{S}}_2^i(\bar\varepsilon) = \widetilde{S}_1^i(\bar\varepsilon)\cap\widetilde{S}_2^i(\bar\varepsilon)$, through the set $\widetilde{S}_1^i(\bar\varepsilon)$, Algorithm 2 computes a new function $h_3(x)$. By choosing a small enough $\bar\varepsilon$ and a large enough γ_3, $\widetilde{S}_3^i(\bar\varepsilon) = \{x : \|\hat g^i(x)^\top \nabla_{x_2} h_j(x)\| \leq \bar\varepsilon\}$ does not intersect $\widetilde{\mathfrak{S}}_2^i(\bar\varepsilon)$.

the intersection of $\bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$ with the hypercuboid around one p_η, for some $\eta \in \{1,\dots,m\}$. By considering the circumscribed hyperellipsoid of the hypercuboid, we conclude that $y_{\ell,2n+1}(t)$ evolves in the intersection of $\bar{\mathcal{C}} \cap \mathrm{Cl}(\mathcal{C}_{v,\mu_v})$ with the closed hyperellipsoid defined by $\sum_{\iota=1}^n \|\hat g^i(y_{\ell,2n+1}(t))^\top \nabla_{x_2} h_\iota(y_{\ell,2n+1}(t))\|^2 \leq 2n\bar\varepsilon^2$, which we denote by $\mathcal{E}(p_\eta,\bar\varepsilon)$.

Let now $x_c := y_{\ell,2n+1}(\mathfrak{l}_{\ell,2n+1}) = y_{\ell,2n}(\mathfrak{r}_{\ell,2n})$, i.e., the first point when $\|\hat g^i(y_{\ell,2n}(t))\nabla_{x_2} h_{2n}(y_{\ell,2n}(t))\| \leq \underline{\varepsilon}$ occurs, where it holds that $\|\hat g^i(x_c)\nabla_{x_2} h_{2n+1}(x_c)\| \geq \gamma_{2n+1}$. Consider a $x \in \mathcal{E}(p_\eta,\bar\varepsilon)$, representing the solution $y_{\ell,2n+1}$. By adding and subtracting $\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)$ to $\|\hat g^i(x_c)\nabla_{x_2} h_{2n+1}(x_c)\|$, we obtain

$$\begin{aligned}&\|\hat g^i(x_c)\nabla_{x_2} h_{2n+1}(x_c) \pm \hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\| \leq \\ &\quad\|\hat g^i(x_c)\nabla_{x_2} h_{2n+1}(x_c) - \hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\| + \\ &\|\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\| \leq \widetilde{L}_{i,2n+1}\|x_c - x\| + \|\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\|,\end{aligned}$$

where $\widetilde{L}_{i,2n+1}$ is the Lipschitz constant of the function $\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)$ in $\mathcal{E}(p_\eta,\bar\varepsilon)$. Since $x, x_c \in \mathcal{E}(p_\eta,\bar\varepsilon)$, there exists a constant $\widetilde{\varepsilon}$ satisfying $\|x_c - x\| \leq \widetilde{\varepsilon}$. By also using $\|\hat g^i(x_c)\nabla_{x_2} h_{2n+1}(x_c)\| \geq \gamma_{2n+1}$, we obtain

$$\gamma_{2n+1} \leq 2\widetilde{L}_{i,2n+1}\widetilde{\varepsilon} + \|\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\|.$$

By choosing $\gamma_{2n+1} \geq \gamma := 2\widetilde{L}_{i,2n+1}\widetilde{\varepsilon} + \underline{\varepsilon} + \chi$, where χ is an arbitrary positive constant, we guarantee that $\|\hat g^i(x)\nabla_{x_2} h_{2n+1}(x)\| \geq \underline{\varepsilon} + \chi$, for all $x \in \mathcal{E}(p_\eta,\bar\varepsilon)$. Therefore, since $\widetilde{\mathfrak{S}}_{2n}^i(\bar\varepsilon) \subset \mathcal{E}(p_\eta,\bar\varepsilon)$, it holds that $\widetilde{\mathfrak{S}}_{2n}^i(\bar\varepsilon) \bigcap \widetilde{S}_{2n+1}^i(\bar\varepsilon) = \emptyset$, implying that the condition of line 4 in Algorithm 2, which would lead to $j = 2n+2$, cannot be satisfied when $j = 2n+1$, leading to the conclusion of the proof.

Proposition 2 is illustrated in Fig. A.1 for a 2-dimensional case.

Proof of Theorem 4: By following the first part of the proof of Theorem 2, we obtain the boundedness of $\beta(h(x_1(t))) \leq \bar\beta$ for a constant $\bar\beta$ and $t \in [t_0, t_{\max})$, implying the boundedness of $x_1(t)$ in a compact set $\widetilde{\mathcal{C}} \subset \mathrm{Int}(\mathcal{C})$, and the boundedness of $x_{2,\mathrm{r}}$, e_2, and $\dot x_{2,r}$ for $t \in [t_0, t_{\max})$. Assume now that $t_{\max}$ is finite and $\lim_{t\to t_{\max}} h_v(x(t)) = 0$, i.e., $\lim_{t\to t_{\max}} \beta_v(h_v(x(t))) = \infty$, which we aim to contradict.

Let $\bar i := i_{t_{\max}} = \max\{i \in \bar{\mathbb{N}} : t_i < t_{\max}\}$, and let $t' := \inf\{t'' \geq t_{\bar i} : x(t) \in \mathcal{C}_{v,\mu_v'}, \forall t \in [t'', t_{\max})\}$. Then it holds that $x(t) \in \mathcal{C}_{v,\mu_v'}$ for all $t \in [t', t_{\max})$, and $x(t) \in \widetilde{\mathcal{C}}_v := \{x \in \mathbb{R}^{2n} : h_v(x) \geq \min_{t\in[t_0,t']}\{h(x(t))\} > 0\}$, for all $t \in [t_0, t']$. Moreover, $\sigma_{\mu_v}(h_v(x(t))) > \sigma_{\mu_v}(\mu_v') > 0$, for all $t \in [t', t_{\max})$. Let the solution restriction $x(t)$, for $t \in [t', t_{\max})$, which can be decomposed, as in (A.5), as

$$x(t) = \begin{cases} y_{1,I_1}(t), & t \in T_{1,1} = [t', \mathfrak{r}_{1,I_1}), \\ \dots \\ y_{1,F_1}(t), & t \in T_{1,F_1} = [\mathfrak{l}_{1,F_1}, \mathfrak{r}_{1,F_1}), \\ y_{2,I_2}(t), & t \in T_{2,I_2} = [\mathfrak{l}_{2,I_2}, \mathfrak{r}_{2,I_2}), \\ \dots \\ y_{2,F_2}(t), & t \in T_{2,F_2} = [\mathfrak{l}_{2,F_2}, \mathfrak{r}_{2,F_2}), \\ \dots \end{cases}$$

with $I_\ell \leq F_\ell \in \{1,\dots,2n+1\}$, for all $\ell \in \{1,\dots,\bar\ell\}$. Moreover, $T_{\ell,j} \subseteq [t', t_{\max})$, $j \in \{I_\ell,\dots,F_\ell\}$. Further, it holds that $f_{j,\mathrm{n}}(x) := \nabla_{x_1} h_j(x)^\top x_2 + \nabla_{x_2} h_j(x)(f(x) + g(x)u_{\mathrm{n}}(x))$ is bounded by a constant $\|f_{j,\mathrm{n}}(y_{\ell,j}(t))\| \leq \bar f_{j,\mathrm{n}}$ due to the continuity of $h_j(\cdot)$, $f(\cdot)$, $g(\cdot)$, $u_{\mathrm{n}}(\cdot)$, and the boundedness of $y(\ell,j)$ for $t \in T_{\ell,j}$. Hence, by using (19), we obtain

$$\dot\beta_j \leq -\kappa_v\sigma_{\mu_v}(\mu_v')(1-\varepsilon)|\beta_{j,\mathrm{d}}| \left(|\beta_{j,\mathrm{d}}| - \frac{\bar f_{j,\mathrm{n}}}{\kappa_v\sigma_{\mu_v}(\mu_v')(1-\varepsilon)}\right),$$

for all $t \in T_{\ell,j}$. Therefore, it holds that $\dot\beta_j < 0$ when $|\beta_{j,\mathrm{d}}| > \sqrt{\frac{\bar f_{j,\mathrm{n}}}{\kappa_v\sigma_{\mu_v}(\mu_v')(1-\varepsilon)}}$. By invoking similar argument as in the proof of Theorem 2, we conclude the boundedness of $\beta_j(h_j(y_{\ell,j}(t))) < \bar\beta_j$ for all $t \in T_{\ell,j}$. At the switching time instants $\mathfrak{l}_{\ell,j} = \mathfrak{r}_{\ell,j-1}$ it holds that $h_j(x(\mathfrak{l}_{\ell,j})) > 0$ and hence the functions β_j are well-defined. Since $h_j(\cdot) < h_v(\cdot)$, we conclude that $\beta_v(y_{\ell,j}(t)) < \bar\beta_j(h_j(y_{\ell,j}(t)))$, for all $t \in T_{\ell,j}$, $j \in \{I_\ell,\dots,F_\ell\}$, which implies that there exists a finite constant $\bar\beta_v$ such that $\beta_v(h_v(x(t))) \leq \bar\beta_v$, for all $t \in [t', t_{\max})$, which contradicts $\lim_{t\to t_{\max}} \beta_v(h_v(x(t))) = \infty$. By also using $x(t) \in \widetilde{\mathcal{C}}_v$, for all $t \in [t_0, t']$ and the compactness of $\widetilde{\mathcal{C}}_v$, we conclude the boundedness of $x(t)$ and $\beta_v(h_v(x(t)))$, for all $t \in [t_0, t_{\max})$. By further invoking [44, Th. 2.1.4], we conclude that $t_{\max} = \infty$, and the boundedness of $x(t)$, $\beta(h(x_1(t)))$, $u(x(t))$ for all $t \in [t_0, \infty)$.