Funnel Control for Uncertain Nonlinear Systems via Zeroing Control Barrier Functions

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***Abstract*— We consider the funnel-control problem for control-affine nonlinear systems with unknown drift term and parametrically uncertain control-input matrix. We develop an adaptive control algorithm that uses zeroing control barrier functions to accomplish trajectory tracking in a pre-defined funnel, achieving hence pre-defined transient and steady-state performance. In contrast to standard funnel-control works, the proposed algorithm can retain the system's input in pre-defined bounds without resorting to reciprocal terms that can lead to arbitrarily large control effort. Moreover and unlike the previous works on zeroing control barrier functions, the algorithm uses appropriately designed adaptation variables that compensate for the uncertainties of the system; namely, the unknown drift term and parametric uncertainty of the control-input matrix. Comparative computer simulations verify the effectiveness of the proposed algorithm.**

***Index Terms*— Funnel control, Prescribed performance control, Barrier functions, Uncertain systems, Adaptive control**

I. INTRODUCTION

During the last decades, a significant research effort has been put towards control of nonlinear systems subject to transient and steady-state constraints. Two large classes of works consist of the so-called Prescribed Performance Control (PPC) [1], [2] and Funnel Control (FC) [3], [4]. In such schemes, the constraints are expressed using time-varying functions that form a funnel. The respective algorithms guarantee then that the system's trajectory tracks a given reference trajectory within the aforementioned funnel, complying thus with the transient and steady-state constraints.

Another significant property in the control of nonlinear systems is robustness against model uncertainties and exogenous disturbances. Typically, dynamical systems entail a large variety of terms that are difficult to model accurately and parameters that cannot be known perfectly a priori. Impressively, PPC and FC methodologies are able to guarantee containment of the tracking error in the pre-defined funnel while at the same time implicitly compensating for large degrees of uncertainty in the nonlinear dynamics [2]–[4]. The cost of doing so, however, is potential application of large control inputs, both in magnitude and rate. More specifically, traditional funnel controllers employ reciprocal barrier functions that diverge to infinity as the tracking error approaches the funnel boundary. In that way, the controller drives the error inside the funnel and compensates the unknown drift terms of the dynamics. However, the discrete nature of the control application in practice might cause the error to approach the funnel boundary, leading to large control inputs that cannot be realized by the system's actuators. The works [5]–[8] accommodate explicit input constraints with sufficiently large bounds, with [7], [8] not resorting to reciprocal terms; [6] further proposes an algorithm using on-the-fly updated funnels, but fails to guarantee that these updated funnels remain bounded. Additionally, standard funnel-based works, including the aforementioned ones, are restricted to systems with the same number of inputs and outputs, assuming a high-gain property that leads to a square and sign-definite input-output matrix.

Except for reciprocal barrier functions, confinement in a given set, such as a funnel, can be established using Control Zeroing Barrier Functions (ZCBF) [9]. In contrast to reciprocal terms, such functions vanish on the boundary of the set and are positive in its interior; the respective control methodologies guarantee then that a ZCBF is non-decreasing on the set boundary, forcing set invariance while complying at the same time with explicit control-input constraints. A significant drawback of ZCBF, however, is the requirement of accurate knowledge of the system model. The recent ZCBF-based works [10], [11] take into account system uncertainty in the form of unknown but constant parameters in the drift term of the control-affine dynamics. Existing ZCBF methodologies cannot accommodate larger degrees of uncertainty in the drift terms or in the control-input matrix.

This paper considers the problem of funnel control for control-affine nonlinear systems with uncertain dynamics. In particular, we consider that the drift term is entirely unknown, whereas the input matrix has known structure but unknown constant parameters. We develop an adaptive control algorithm that achieves tracking of a reference trajectory in a pre-defined funnel by using Zeroing Control Barrier Functions. The algorithm further uses appropriately defined adaptation variables that compensate for the system uncertainties by assuming known upper bounds of the unknown parameters of the input matrix and of the drift term when the system evolves inside the funnel. Compared to previous funnel-based works (e.g., [1]–[5]), the proposed algorithm i) avoids reciprocal terms that might yield excessively large control inputs, ii) can handle systems with fewer inputs than outputs, and iii) efficiently accommodates explicit input constraints. In addition, in contrast to previous ZCBF-based works, which considered

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only parametric uncertainty in the drift term, we consider parametric uncertainty in the input matrix and further that the drift term is unknown.

Notation: We denote the sets of nonnegative and positive reals by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively. Given a function $h : \mathbb{R}^n \to \mathbb{R}^p$, we use $\nabla h(x) = \frac{\mathrm{d}h(x)}{\mathrm{d}x} \in \mathbb{R}^{p \times n}$; $\mathrm{Cl}(\cdot)$ denotes the closure of a set. A continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$, with $a > 0$, is a class-$\mathcal{K}$ function if $\alpha(0) = 0$ and α is strictly monotonically increasing. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, then α is said to be a class-$\mathcal{K}_\infty$ function. A continuous function $\alpha : (-b, a) \to \mathbb{R}_{\geq 0}$, with $a, b > 0$, is an *extended class-$\mathcal{K}$ function* if $\alpha(0) = 0$ and α is strictly monotonically increasing. If $a, b = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, then α is said to be an *extended* class-$\mathcal{K}_\infty$ function. A continuous $\beta : [0, b) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $b > 0$, is a $\mathcal{KL}$ function if, for each fixed s, $\beta(r, s)$ is a class-$\mathcal{K}$ function with respect to r and, for each fixed r, $\beta(r, s)$ is decreasing with respect to s and $\lim_{s\to\infty} \beta(r, s) = 0$. Finally, we define $\mathcal{N} \coloneqq \{1, \dots, n\}$, $\mathcal{M} \coloneqq \{1, \dots, m\}$, $\mathcal{P} \coloneqq \{1, \dots, p\}$, $\mathcal{L} \coloneqq \{1, \dots, \ell\}$.

II. Preliminaries

A. Zeroing Control Barrier Functions

Consider a system of the form $\dot{x} = f(x) + g(x)u$, where $f(\cdot)$ and $g(\cdot)$ are locally Lipschitz functions and u is constrained in a compact set $U \subset \mathbb{R}^m$.

Definition 1: [9] Let $\mathcal{C} \subset D$ be the superlevel set of a continuously differentiable function $B : D \subset \mathbb{R}^n \to \mathbb{R}$, i.e., $\mathcal{C} \coloneqq \{x \in \mathbb{R}^n : B(x) \geq 0\}$. Then, B is a zeroing control-barrier function (ZCBF) for the system $\dot{x} = f(x) + g(x)u$ if there exists an extended class-$\mathcal{K}_\infty$ function α such that,

$$\sup_{u\in U} \left[\nabla B(x)^\top (f(x) + g(x)u) + \alpha(B(x))\right] \geq 0, x \in D \quad (1)$$

Following [9], one can define $K(x) \coloneqq \{u \in U : \nabla B(x)^\top (f(x) + g(x)u) + \alpha(B(x)) \geq 0\}$. Then, any locally Lipschitz function $u : D \to U$ that satisfies $u(x) \in K(x)$ for all $x \in D$ guarantees that $x(t) \in \mathcal{C}$, for all $t \in [t_0, t_{\max})$, given that $x(t_0) \in \mathcal{C}$, where $[t_0, t_{\max})$ is the maximal interval of existence of the solution of $\dot{x} = f(x) + g(x)u(x)$.

B. Projection Operator

The projection operator is an adaptive-control technique that guarantees the evolution of estimates of unknown parameters of the system in a priori known sets [12]. Let a constant vector $q^* \in \Pi \subset \mathbb{R}^\ell$ and a real function $p : \Pi \to \mathbb{R}$ satisfying $p(q^*) \leq 0$. Further assume that the set $\{q \in \mathbb{R}^\ell : p(q) \leq \lambda\}$ is convex and contained in Π for each $\lambda \in [0, 1]$ and that $\nabla p(q)$ is nonzero for all q satisfying $p(q) \in [0, 1]$. Let a time-varying estimate $\hat{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ of q^* satisfying $\hat{q}(0) \in \Pi_c \coloneqq \{q \in \mathbb{R}^\ell : p(q) \leq 1\}$ and evolving according to $\dot{\hat{q}} = \mathrm{Proj}(y, p)$, where

$$\mathrm{Proj}(y, p) \coloneqq \begin{cases} y, & \text{if } p \leq 0 \text{ or } (p \geq 0 \text{ and } \nabla p^\top y \leq 0) \\ \left(I - \frac{p \nabla p \nabla p^\top}{\nabla p^\top \nabla p}\right) y, & \text{otherwise} \end{cases} \quad (2)$$

and $y : [0, \infty) \to \mathbb{R}^\ell$ is a continuous function. Then [12] 1) $\hat{q}(t) \in \Pi_c$, for all $t \geq 0$; 2) $(q^* - \hat{q})^\top \mathrm{Proj}(y, p) \geq (q^* - \hat{q})^\top y$; and 3) $\mathrm{Proj}(y, p)$ is Lipschitz continuous.

C. Comparison principle

We provide a lemma on differential inequalities that will be used later in the proof of the main result of the paper.

Lemma 1: Let $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz continuous extended class-$\mathcal{K}_\infty$ function and $\eta : [t_0, t_{\max}) \to \mathbb{R}$ a continuously differentiable function, with $t_{\max} > t_0 \geq 0$. If $\eta(t_0) \geq 0$ and $\dot{\eta}(t) \geq -\alpha(\eta(t))$, for all $t \in [t_0, t_{\max})$, then $\eta(t) \geq 0$, for all $t \in [t_0, t_{\max})$.

Proof: Consider the differential equation $\dot{z}(t) = -\alpha(z(t))$, $z(t_0) = \eta(t_0)$. Since $z(t_0) \geq 0$ and the restriction of α to $\mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function, there exists a unique solution $z(t) = \sigma(z(t_0), t)$, $t \geq t_0$, where $\sigma()$ is a class-$\mathcal{KL}$ function [13, Lemma 4.4]. Since $\dot{\eta}(t) \geq -\alpha(\eta(t))$ for $t \in [t_0, t_{\max})$, one can prove by using the Comparison Lemma [13, Lemma 3.4] that $\eta(t) \geq \sigma(h(t_0), t)$ for all $t \in [t_0, t_{\max})$. Since $\sigma(\cdot)$ is a class-$\mathcal{KL}$ function, it holds that $\eta(t) \geq \sigma(h(t_0), t) \geq 0$, $t \in [t_0, t_{\max})$. ■

III. Problem Formulation

We consider nonlinear systems of the form

$$\dot{x} = f(x, t) + g(x, t)u \quad (3a)$$
$$y = h(x) \quad (3b)$$

where $x = [x_1, \dots, x_n] \in \mathbb{R}^n$ and $y = [y_1, \dots, y_p] \in \mathbb{R}^p$ are the system's state and output, respectively, which are available for measurement, and u is the control input restricted in a compact set $U \subset \mathbb{R}^m$; $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$ are functions that are continuous in t and locally Lipschitz in x, and $h = [h_1, \dots, h_p] : \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function with radially unbounded components, i.e., $\lim_{\|x\|\to\infty} |h_i(x)| = \infty$, for all $i \in \mathcal{P}$. We further require $\nabla h(x)^\top g(x, t)$ to not be identically zero, i.e. the system has output-relative degree one. We assume that $f(x, \cdot)$ and $g(x, \cdot)$ are uniformly bounded for each fixed $x \in \mathbb{R}^n$. We consider that the drift term $f(\cdot)$ is *unknown*, while $g(\cdot)$ is linearly parametrized by a vector of constant unknown weights θ^* as $g(x, t) = g(x, t, \theta^*) = [\Delta_{ij}(x, t)^\top \theta^*]_{i\in\mathcal{N}, j\in\mathcal{M}}$, where $\Delta_{ij}(x, t) \in \mathbb{R}^\ell$, $i \in \mathcal{N}, j \in \mathcal{M}$ are known functions. Therefore, it holds that

$$g(x, t, \theta^*)u = \begin{bmatrix} \sum_{j\in\mathcal{M}} \Delta_{1j}(x, t)^\top u_j \\ \vdots \\ \sum_{j\in\mathcal{M}} \Delta_{nj}(x, t)^\top u_j \end{bmatrix} \theta^* =: \widetilde{g}(x, t, u)\theta^* \quad (4)$$

which will be used later. Finally, the boundedness of $f(x, t)$ in t implies that $\|f(x, t)\| \leq f_B(x)$, for all $t \geq 0$, where $f_B : \mathbb{R}^n \to \mathbb{R}^n$ is an unknown function.

The control objective in this paper is the design of a control algorithm u such that the output y tracks a smooth and bounded reference trajectory $y_\mathrm{d} = [y_{\mathrm{d},1}, \dots, y_{\mathrm{d},p}] : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, with bounded derivatives, within a pre-defined funnel. More specifically, given p funnels described by the smooth functions $\rho_i : \mathbb{R}_{\geq 0} \to [\underline{\rho}_i, \bar{\rho}_i] \subset \mathbb{R}_{>0}$, where $\underline{\rho}_i$ and $\bar{\rho}_i$ are positive lower and upper bounds, respectively, we aim at guaranteeing that $-\rho_i(t) < y_i(t) - y_{\mathrm{d},i}(t) < \rho_i(t)$ for all $t \geq 0$ and $i \in \mathcal{P}$, provided that $-\rho_i(0) < y_i(0) - y_{\mathrm{d},i}(0) < \rho_i(0)$, $i \in \mathcal{P}$, i.e., initial compliance with the funnels. Therefore, the objective is the design of $u \in U$ such that

$$x(t) \in \Omega(t) \coloneqq \{x \in \mathbb{R}^n : -\rho_i(t) < h_i(x) - y_{\mathrm{d},i}(t) < \rho_i(t), \forall i \in \mathcal{P}\}, \quad (5)$$

for all $t \geq 0$. Note that $\Omega(t)$ is bounded for each $t \geq 0$ since $y_{\rm d}(t)$ is bounded and $h(x)$ is continuously differentiable with radially unbounded components. We further define the positive constants $\bar{f} := \sup_{x\in\Omega(t),t\geq 0}\{f_B(x)\}$, $\bar{\rho}(t) := \frac{1}{\min_{i\in\mathcal{P}}\{\rho_i(t)\}}$ to be used in the sequel. To solve the aforementioned problem, we require the following assumption.

Assumption 1: There exists a known positive constant F and a known convex open set $\Pi_\theta \subset \mathbb{R}^\ell$ satisfying $F \geq \bar{f} = \sup_{x\in\Omega(t),t\geq 0}\{f_B(x)\}$ and $\theta^* \in \Pi_\theta$.

Assumption 1 provides known bounds for $\bar{f}$ and θ^* when $y(t) - y_{\rm d}(t)$ evolves in the prescribed funnels. The drift term $f(x,t)$ usually depends on parameters of the system, such as masses or moments of inertia, and exogenous time-varying disturbances. One can obtain upper-bound estimates for these parameters, either via the manufacturer specifications or via experimental identification. For the time-varying disturbances, one can obtain upper-bound estimates via experimentation or evaluation of environmental conditions (e.g., wind or ocean currents for aerial and underwater vehicles, respectively) for the desired region of operation, dictated by $\rho_i(t)$ and $\mathrm{y}_d(t)$. Similarly, $g(\cdot)$ represents the inertia of the system and its structure can be derived using standard physical laws; θ consists of parameters of the system whose upper bounds can be estimated, as with $\bar{f}$. One can also use data-driven techniques to acquire the aforementioned estimates (e.g. [14], which computes over-approximations of $f(\cdot)$ and $g(\cdot)$). Nevertheless, Assumption 1 can be relaxed, as we describe in the next section.

IV. MAIN RESULTS

This section presents the main results of this paper. We design an adaptive control algorithm based on ZCBFs to guarantee evolution of the error $e := y - y_{\rm d}$ in the prescribed funnels despite the uncertainties θ^* in $g(x,t)$ and the unknown term $f(x,t)$. We begin by defining the error

$$\xi = \xi_f(x,t) := \rho(t)^{-1}e = \rho(t)^{-1}(h(x) - y_{\rm d}(t))$$

where $\rho := \mathrm{diag}\{\rho_i\}_{i\in\mathcal{P}}$. The control objective is equivalent to maintaining the normalized error $\xi(t)$ in the set $(-1,1)^p$. To do so, we define the continuously differentiable barrier function $b : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and its 0-superlevel set as

$$b(x,t) := \frac{1}{2}(1 - \|\xi\|^2) = \frac{1}{2}(1 - \|\xi_f(x,t)\|^2)$$
$$\mathcal{C}(t) := \{x \in \mathbb{R}^n : b(x,t) \geq 0\} = \mathrm{Cl}(\Omega(t)).$$

We note that asymmetric funnel constraints of the form $-M_i\rho_i(t) \leq e_i(t) \leq \rho_i(t)$, with $M_i \in [0,1)$, $i \in \mathcal{P}$, can be accommodated by setting $b(x,t) = \frac{1}{2}(1 - (\xi - \xi_0)^\top M(\xi - \xi_0))$, where $\xi_0 := \frac{1}{2}[1 + M_1, \dots, 1 + M_p]^\top$ and $M := 2\mathrm{diag}\{[1 - M_i]^{-1}_{i\in\mathcal{P}}\}$. The goal is to render the set $\Omega(t)$ forward invariant, i.e., to guarantee that $x(t) \in \Omega(t)$ for all $t > 0$, given that $x(0) \in \Omega(0)$ [9]. By evaluating the derivative of $b(x,t)$ along (3), one obtains $\frac{\partial b(x,t)}{\partial x}^\top \dot{x} + \frac{\partial b(x,t)}{\partial t} = -\xi^\top\rho^{-1}(\nabla h(x)^\top(f(x,t) + g(x,t)u) - y_{\rm d}(t) - \dot{\rho}\xi)$, which reveals a controllability loss when $\xi = 0$. Therefore, for the purposes of control design, we are interested in the case when $\|\xi\| \geq \delta$ for a constant $\delta \in (0,1)$, which also leads to the definition of the set

$$\mathcal{C}^\delta(t) = \{x \in \mathbb{R}^n : \frac{1}{2}(1 - \delta^2) \geq b(x,t) \geq 0\} \tag{6}$$

Similar to [10], [11], the unknown parameters θ^* and drift term $f(\cdot)$ lead us to pursue forward invariance of a set more tightened than $\mathcal{C}(t)$. We first define estimates for $\bar{f}$ and θ^* as $\hat{f} \in \mathbb{R}$ and $\hat{\theta} = [\hat{\theta}_1, \dots, \hat{\theta}_\ell] \in \mathbb{R}^\ell$, respectively, and the associated errors $\widetilde{f} := \bar{f} - \hat{f}$ and $\widetilde{\theta} = [\widetilde{\theta}_1, \dots, \widetilde{\theta}_\ell] := \theta^* - \hat{\theta}$. In view of Assumption 1 and Section II-B, we will design adaptation laws $\dot{\hat{\theta}}$ and $\dot{\hat{f}}$ such that $\hat{\theta}$ and $\hat{f}$ evolve in predefined sets $\Pi_{c,\theta}$ and $\Pi_{c,f}$. To that end, we design functions $p_\theta : \Pi_\theta \to \mathbb{R}$, $p_f : (0,F] \to \mathbb{R}$ such that [12]
1) for each $\lambda \in [0,1]$, the sets $\{q \in \mathbb{R}^\ell : p_\theta(q) \leq \lambda\}$ and $\{q \in \mathbb{R} : p_f(q) \leq \lambda\}$ are convex and contained in Π_θ and $(0,F]$, respectively;
2) $\nabla p_\theta(\cdot)$ and $\nabla p_f(\cdot)$ are nonzero in the sets $\{q \in \mathbb{R}^\ell : p_\theta(q) \in [0,1]\}$ and $\{q \in \mathbb{R} : p_f(q) \in [0,1]\}$, respectively;
3) $p_\theta(\theta^*) \leq 0$ and $p_f(\bar{f}) \leq 0$.
We define then $\Pi_{c,\theta} := \{q \in \mathbb{R}^\ell : p_\theta(q) \leq 1\}$ and $\Pi_{c,f} := \{q \in \mathbb{R} : p_f(q) \leq 1\}$ and $B_f := \max\{\|\bar{f} - \hat{f}\| : \bar{f} \in [0,F], \hat{f} \in \Pi_{c,f}\}$, $B_\theta := \max\{\|\theta^* - \hat{\theta}\| : \theta^* \in \Pi_\theta, \hat{\theta} \in \Pi_{c,\theta}\}$. Note that $\hat{\theta} \in \Pi_{c,\theta}$ and $\hat{f} \in \Pi_{c,f}$ imply that $\|\widetilde{\theta}\| \leq B_\theta$ and $|\widetilde{f}| \leq B_f$. We can now define the time-varying set $\mathcal{C}_{\hat{\theta},\hat{f}}(t)$, parametrized by $\hat{\theta}$, $\hat{f}$, as

$$\mathcal{C}_{\hat{\theta},\hat{f}}(t) = \left\{x \in \mathbb{R}^n : b(x,t) \geq \frac{1}{2k_\theta}\|\widetilde{\theta}\|^2 + \frac{1}{2k_f}\widetilde{f}^2\right\}$$

and, in view of (6), the "local" set $\mathcal{C}^\delta_{\hat{\theta},\hat{f}}(t) := \mathcal{C}_{\hat{\theta},\hat{f}}(t) \bigcap \{x \in \mathbb{R}^n : \|\xi_f(x,t)\| \geq \delta\} \subset \mathcal{C}_{\hat{\theta},\hat{f}}(t)$, i.e.,

$$\mathcal{C}^\delta_{\hat{\theta},\hat{f}}(t) := \left\{x \in \mathbb{R}^n : \frac{1}{2}(1 - \delta^2) \geq b(x,t) \geq \frac{1}{2k_\theta}\|\widetilde{\theta}\|^2 + \frac{1}{2k_f}\widetilde{f}^2\right\}$$

for a given $\delta \in (0,1)$, where k_θ and k_f are positive constants that dictate how tightened $\mathcal{C}_{\hat{\theta},\hat{f}}(t)$ and $\mathcal{C}^\delta_{\hat{\theta},\hat{f}}(t)$ are and will be chosen later to guarantee that $x(0) \in \mathcal{C}_{\hat{\theta}(0),\hat{f}(0)}(0)$. We further define $\mathbb{D}^\delta := \mathcal{C}^\delta_{\hat{\theta},\hat{f}}(t) \times \mathbb{R}_{\geq 0} \times \Pi_{c,\theta} \times \Pi_{c,f}$, for a constant $\delta \in (0,1)$, and give the definition of adaptive zeroing control barrier functions for funnel control, which is associated with the non-emptiness of the set

$$\begin{aligned}K_{u,\alpha}(x,t,\hat{\theta},\hat{f}) := \Bigg\{ & u \in U : -\bar{\rho}(t)\|\nabla h(x)\|\hat{f}\|\xi\| \\ & - \xi^\top\rho(t)^{-1}\left(\nabla h(x)^\top g(x,t,\hat{\theta})u - \dot{y}_{\rm d}(t) - \dot{\rho}(t)\xi\right) \\ & \geq -\alpha\left(b(x,t) - \frac{1}{2k_\theta}B_\theta^2 - \frac{1}{2k_f}B_f^2\right)\Bigg\}\end{aligned} \tag{7}$$

for a class-$\mathcal{K}_\infty$ function $\alpha(\cdot)$.

Definition 2: The function $b(x,t) = \frac{1}{2}(1 - \|\xi\|^2)$ is a δ-funnel-control adaptive zeroing control barrier function (δ-FaZCBF) for (3) if there exists a locally Lipschitz extended class-$\mathcal{K}_\infty$ function α such that $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ is non-empty for all $(x,t,\hat{\theta},\hat{f}) \in \mathbb{D}^\delta$.

The non-emptiness of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ is reminiscent of the standard ZCBF-based conditions, such as (1) and the ones in

[9]. The first term corresponds to a lower bound of the term $-\xi^\top \rho(t)^{-1}\nabla h(x)^\top f(x,t)$ when $x(t)\in\Omega(t)$ and accommodates the unknown drift term $f(x,t)$. Note that, since the terms $\widetilde{\theta}$ and $\widetilde{f}$ are unknown, we cannot use them in the definition of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ in order to enforce forward invariance of $\mathcal{C}(t)$. Therefore, we use the more conservative estimates B_θ and B_f; the subsequent control design guarantees, along with the known bounds from Assumption 1, that $\|\widetilde{\theta}\|\le B_\theta$ and $|\widetilde{f}|\le B_f$ and hence $-\alpha(b(x,t)-\frac{1}{2k_\theta}B_\theta^2-\frac{1}{2k_f}B_f^2)\ge-\alpha(b(x,t))$, since α is an extended class-$\mathcal{K}_\infty$ function.

The control design consists of computing a controller that satisfies $u(x,t,\hat{\theta},\hat{f})\in K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ for all $(x,t,\hat{\theta},\hat{f})\in\mathbb{D}^\delta$, given a δ-FaZCBF $b(x,t)$. Similarly to previous works on ZCBFs, we use the quadratic program (QP)

$$\begin{aligned} u^*(x,t,\hat{\theta},\hat{f}) = &\operatorname*{arg\,min}_{u\in U}\|u\|^2 \qquad (8)\\ \text{s.t.}\ &-\bar{\rho}(t)\|\nabla h(x)\|\hat{f}\|\xi\|-\xi^\top\rho(t)^{-1}(\nabla h(x)^\top g(x,t,\hat{\theta})u\\ &-\dot{y}_{\mathrm{d}}(t)-\dot{\rho}(t)\xi)\ge-\alpha(b(x,t)-\frac{1}{2k_\theta}B_\theta^2-\frac{1}{2k_f}B_f^2)\end{aligned}$$

Given a function $b(x,t)$, the feasibility of (8) for $(x,t,\hat{\theta},\hat{f})\in\mathbb{D}^\delta$ guarantees that $b(x,t)$ is a δ-FaZCBF. Further note that the QP has polynomial-time complexity [15] and can be hence efficiently used for high-dimensional systems.

In view of the definition of $\mathcal{C}^\delta_{\hat{\theta},\hat{f}}(t)$, we compute the control input u via the aforementioned QP only for $\|\xi\|\ge\delta$. To that end, we design a procedure that "activates" the control u only when $\|\xi(t)\|\in(\delta,1)$, with $0<\delta<1$. We define hence a smooth switching function $\phi:\mathbb{R}\to[0,1]$, as in [16], as $\phi(q)=0$ if $q<\delta$, $\phi(q)=\kappa(q)$ if $\delta\le q\le\delta_2$, and $\phi(q)=1$ if $q\ge\delta_2$, where δ_2 is a positive constant satisfying $\delta_2\in(\delta,1)$ and $\kappa:\mathbb{R}\to[0,1]$ is a locally Lipschitz function with $\kappa(\delta)=0$ and $\kappa(\delta_2)=1$. When $\|\xi\|<\delta$, i.e., when $y(t)$ is sufficiently close to $y_{\mathrm{d}}(t)$, one can possibly employ a pre-defined nominal controller $u_{\mathrm{n}}(x,t)$, locally Lipschitz continuous in x and piecewise continuous in t, which aims to achieve a secondary control objective.

The main results of this paper are given in the next theorem, which shows that the existence of a δ-FaZCBF, coupled with appropriately designed adaptation laws, renders the set $\Omega(t)$ forward invariant.

Theorem 1: Let $b:\mathbb{R}^n\times\mathbb{R}_{\ge0}$ be a δ-FaZCBF for (3) and a constant $\delta\in(0,1)$. Let $u_s:\mathbb{D}^\delta\to U$ be the solution of (8) and $u_{\mathrm{n}}(x,t):\mathbb{R}^n\times\mathbb{R}_{\ge0}\to U$ be a map locally Lipschitz in x and piecewise continuous in t. Under Assumption 1, if δ_2, k_f, and k_θ satisfy

$$1>\max\{\delta_2,\|\xi(0)\|\}^2+\frac{1}{k_\theta}B_\theta^2+\frac{1}{k_f}B_f^2 \qquad (9)$$

then the adaptive control law

$$u(x,t,\hat{\theta},\hat{f})=\phi(\|\xi\|)u_s(x,\hat{f},\hat{\theta},t)+(1-\phi(\|\xi\|)u_{\mathrm{n}}(x,t) \qquad (10a)$$

$$\dot{\hat{\theta}}=k_\theta\mathrm{Proj}\bigg(\widetilde{g}(x,t,u)^\top\nabla h(x)\rho(t)^{-1}\xi,p_\theta(\hat{\theta})\bigg) \qquad (10b)$$

$$\dot{\hat{f}}=k_f\mathrm{Proj}\bigg(\bar{\rho}(t)\|\nabla h(x)\|\|\xi\|,p_f(\hat{f})\bigg) \qquad (10c)$$

renders the set $\Omega(t)$ forward invariant.

Proof: We note first that, according to the properties of the projection operator (2), the right-hand side of (10b) and (10c) is locally Lipschitz continuous. Additionally, the solution of (8) is locally Lipschitz in $(x,\hat{\theta},\hat{f})$ and piecewise continuous in t [9]. Therefore, the closed-loop system formed by (3) and (10) has a local continuously differentiable solution $x(t)$, $\hat{\theta}(t)$, $\hat{f}(t)$ defined on a maximal time interval $T\coloneqq[0,t_{\max})$, with $t_{\max}>0$. Next, in view of the first property of the projection operator (2), the adaptation laws (10b) and (10c) guarantee that $\hat{\theta}(t)\in\Pi_{c,\theta}$ and $\hat{f}(t)\in\Pi_{c,f}$; since $\theta^*\in\Pi_\theta$ and $\bar{f}\in(0,F]$, we conclude that $\|\widetilde{\theta}(t)\|\le B_\theta$ and $|\widetilde{f}(t)|\le B_f$ for all $t\in T$.

Next, note that (9) guarantees that $\mathcal{D}(t)\coloneqq\{x\in\mathbb{R}^n:\|\xi\|\le\delta_2\}\subseteq\mathcal{C}_{\hat{\theta}(t),\hat{f}(t)}(t)$, $t\in T$, and that $x(0)\in\mathcal{C}_{\hat{\theta}(0),\hat{f}(0)}(0)\subset\mathcal{C}(0)$. The continuity of the solution implies that we can choose $T'\coloneqq[0,t'_{\max})\subseteq T$ such that $x(t)\in\mathcal{C}_{\hat{\theta}(t),\hat{f}(t)}(t)\subset\mathcal{C}(t)$, for all $t\in T'$. We define next

$$b_2\coloneqq b(x,t)-\frac{1}{2k_\theta}\|\widetilde{\theta}\|^2-\frac{1}{2k_f}\widetilde{f}^2 \qquad (11)$$

and restrict the analysis for when $\|\xi\|\ge\delta_2>\delta$, i.e., the time intervals $T^{\delta_2}\subset T'$ such that $(x(t),t,\hat{\theta}(t),\hat{f}(t))\in\mathbb{D}^{\delta_2}$, for all $t\in T^{\delta_2}$. Note that $b_2(x(t),t)\ge0$ for all $t\in T'\backslash T^{\delta_2}$ since $\mathcal{D}(t)\subseteq\mathcal{C}_{\hat{\theta}(t),\hat{f}(t)}(t)$ and that $b_2(x(t_I),t_I)\ge1-\|\xi(t_I)\|^2-\frac{1}{2k_\theta}B_\theta^2-\frac{1}{2k_f}B_f^2\ge0$ for t_I satisfying $\|\xi(t_I)\|=\delta_2$.

Differentiation of b_2 in T^{δ_2} along the solution of the closed-loop system and use of $-\xi^\top\rho(t)^{-1}\nabla h(x)^\top f(x,t)\ge-\bar{\rho}(t)\|\nabla h(x)\|\bar{f}\|\xi\|$ and $\widetilde{\theta}=\theta^*-\hat{\theta}$, $\widetilde{f}=\bar{f}-\hat{f}$, yields

$$\begin{aligned}\dot{b}_2\ge&-\bar{\rho}(t)\|\nabla h(x)\|\hat{f}\|\xi\|-\xi^\top\rho(t)^{-1}\nabla h(x)^\top g(x,t,\hat{\theta})u\\ &+\xi^\top\rho(t)^{-1}(\dot{y}_{\mathrm{d}}(t)+\dot{\rho}(t)\xi)-\bar{\rho}(t)\|\nabla h(x)\|\widetilde{f}\|\xi\|\\ &-\xi^\top\rho(t)^{-1}\nabla h(x)^\top\widetilde{g}(x,t,u)\widetilde{\theta}+\frac{1}{k_\theta}\widetilde{\theta}^\top\dot{\hat{\theta}}+\frac{1}{k_f}\widetilde{f}\dot{\hat{f}}\end{aligned}$$

for all $t\in T^{\delta_2}$. By using (10b), (10c), and the second property of the projection operator (2), $\dot{b}_2$ becomes

$$\begin{aligned}\dot{b}_2\ge&-\bar{\rho}(t)\|\nabla h(x)\|\hat{f}\|\xi\|-\xi^\top\rho(t)^{-1}\nabla h(x)^\top g(x,t,\hat{\theta})u\\ &+\xi^\top\rho(t)^{-1}(\dot{y}_{\mathrm{d}}(t)+\dot{\rho}\xi)\end{aligned}$$

Since $\|\xi(t)\|\ge\delta_2$ for all $t\in T^{\delta_2}$, it holds that $u=u_s(x,t,\hat{\theta},\hat{f})$. Moreover, since b is a δ-FaZCBF for (3) and $\mathbb{D}^{\delta_2}\subset\mathbb{D}^\delta$, $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ is non-empty and $u=u_s(x(t),t,\hat{\theta}(t),\hat{f}(t))\in K_{u,\alpha}(x(t),t,\hat{\theta}(t),\hat{f}(t))$ for all $t\in T^{\delta_2}$. Hence, by using (7), we obtain $\dot{b}_2\ge-\alpha(b(x(t),t)-\frac{1}{2k_\theta}B_\theta^2-\frac{1}{2k_f}B_f^2)$ for all $t\in T^{\delta_2}$. Since α is an extended class-$\mathcal{K}_\infty$ function and $\|\widetilde{\theta}\|\le B_\theta$, $\|\widetilde{f}\|\le B_f$, we conclude that $\dot{b}_2\ge\alpha(b_2(x(t),t)$. By applying Lemma 1 for the intervals T^{δ_2}, we conclude that $b_2(x(t),t)\ge0$ and hence $b(x(t),t)>b_2(x(t),t)\ge0$ for all $t\in T^{\delta_2}$. Moreover, it holds that $b_2(x(t),t)\ge0$ for all $t\in T'\backslash T^{\delta_2}$, from which we conclude that $b_2(x(t),t)\ge0$ for all $t\in[0,t'_{\max})$. Consequently, it holds that $x(t)\in\mathcal{C}_{\hat{\theta}(t),\hat{f}(t)}(t)\subset\mathcal{C}(t)$ for all $[t_I,t'_{\max})$. Since $\mathcal{C}(t)$ is compact, we conclude that $T'=T=[0,\infty)$, rendering $\mathcal{C}(t)$ forward invariant. Since $\mathcal{C}_{\hat{\theta}(t),\hat{f}(t)}(t)\subset\mathcal{C}(t)$ and the results hold for any $\xi(0)\in\Omega(0)$ owing to (9), $\Omega(t)$ is forward invariant. ■

Remark 1: The QP (8) provides sufficient feasibility conditions for the non-emptiness of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ in $\mathbb{D}^\delta$ and the verification that $b(x,t)$ is a δ-FaZCBF; the function that minimizes $\|u\|^2$ subject to the constraint of (8) is [9]

$$u^*(x,t,\hat{\theta},\hat{f}) = \begin{cases} 0, & \text{if } \psi_0(x,t,\hat{f}) \geq 0 \\ -\dfrac{\psi_0(x,t,\hat{f})\psi_1(x,t,\hat{\theta})^\top}{\psi_1(x,t,\hat{f})\psi_1(x,t,\hat{\theta})^\top}, & \text{if } \psi_0(x,t,\hat{f}) < 0 \end{cases}$$

where $\psi_0(x,t,\hat{f}) := -\bar{\rho}(t)\|\nabla h(x)\|\hat{f}\|\xi\| + \xi^\top \rho(t)^{-1}(\dot{y}_\text{d}(t) + \dot{\rho}(t)\xi) + \alpha\left(b(x,t) - \frac{1}{2k_\theta}B_\theta^2 - \frac{1}{2k_f}B_f^2\right)$ and $\psi_1(x,t,\hat{\theta}) := -\xi^\top\rho(t)^{-1}\nabla h(x)^\top g(x,t,\hat{\theta})$. Therefore, the feasibility of (8), and hence the non-emptiness of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ in $\mathbb{D}^\delta$, requires $u^*(x,t,\hat{\theta},\hat{f}) \in U$, for all $(x,t,\hat{\theta},\hat{f}) \in \mathbb{D}^\delta$. Hence, a sufficient condition for the non-emptiness of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ is $\sup_{(x,t,\hat{\theta},\hat{f})\in\mathbb{D}^\delta} \frac{|\psi_0(x,t,\hat{f})|}{\|\psi_1(x,t,\hat{\theta})\|} \leq \bar{U}$, where $\bar{U}$ is the maximum of the set U. Similar conditions are required in ZCBF works (e.g., [9]–[11]) as well as funnel-based works with explicit input constraints [5]–[7]. Note that such a condition requires the positivity of $\|\psi_1(x,t,\hat{\theta})\| = \|\xi^\top\rho(t)^{-1}\nabla h(x)^\top g(x,t,\hat{\theta})\|$, which, in view of (8), can be viewed as a sufficiently controllability condition. Intuitively, $\mathbb{D}^\delta$ must exclude singular configurations in the nullspace of $\nabla h(x)^\top g(x,t,\hat{\theta})$. In case such a condition is not satisfied, one can employ additional barrier functions of the form $h_\mathcal{S}(x,t,\hat{\theta}) = \|\xi^\top\rho(t)^{-1}\nabla h(x)^\top g(x,t,\hat{\theta})\|^2 - \underline{c}$ for a positive constant $\underline{c}$; such analysis, however, falls out of the scope of this paper. Note that standard funnel-based works consider systems with equal number of inputs and outputs and assume a square and sign-definite high-gain matrix of the form $\nabla h(x)^\top g(x,t)$ [1]–[7]. Finally, note that the aforementioned condition on the non-emptiness of $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ in $\mathbb{D}^\delta$ is sufficient, but not necessary, for the funnel-control objective. That is, given an initial configuration $(x(0),\hat{\theta}(0),\hat{f}(0))$, it is only required that $K_{u,\alpha}(x(t),t,\hat{\theta}(t),\hat{f}(t))$ is non-empty along the solution $(x(t),\hat{\theta}(t),\hat{f}(t))$ of the closed-loop system from the initial configuration. Therefore, as also demonstrated in Section V, the proposed algorithm can achieve funnel containment even if $K_{u,\alpha}(x,t,\hat{\theta},\hat{f})$ is empty in some parts of $\mathbb{D}^\delta$.

Remark 2: By inspecting the proof of Theorem 1, one can conclude that the proposed algorithm guarantees forward invariance of $\mathcal{C}_{\hat{\theta},\hat{f}}(t)$. This is achieved *without using* the unknown $\widetilde{\theta}$ and $\widetilde{f}$; the algorithm only uses the upper bounds B_θ and B_f, which require knowledge of the upper bounds of $\bar{f}$ and θ^* (Assumption 1). We stress that, if such bounds are unknown, one can still use the proposed algorithm by replacing $-\alpha(b(x,t) - \frac{1}{2k_\theta}B_\theta^2 - \frac{1}{2k_f}B_f^2)$ with 0 in (8), and the adaptation laws in (10) with $\dot{\hat{\theta}} = k_\theta\widetilde{g}(x,t,u)^\top\nabla h(x)\rho(t)^{-1}\xi$, $\dot{\hat{f}} = k_f\bar{\rho}(t)\|\nabla h(x)\|\|\xi\|$. Similarly to the proof of Theorem 1, one can show in that case that $\dot{b}_2(x,t) \geq 0$ for all $(x(t),t,\hat{\theta},\hat{f}) \in \mathbb{D}^{\delta_2}$, with b_2 defined in (11), and consequently, that $b(x(t),t) \geq 0$ for all $t \geq 0$. However, the condition $\dot{b}_2(x(t),t) \geq 0$ is more conservative than $\dot{b}_2(x(t),t) \geq -\alpha(b_2(x(t),t))$, which is achieved in the proof of Theorem 1; $\dot{b}_2(x(t),t) \geq 0$ implies that all the level sets of b_2 are forward invariant when $\|\xi\| \geq \delta_2$, forcing thus the system to evolve in a subset of $C_{\hat{\theta},\hat{f}}(t)$. Additionally, if bounds for $\bar{f}$ and θ^* are unknown, one cannot retain $\hat{\theta}(t)$ and $\hat{f}(t)$ in a priori known sets, which prevents the derivation of an a priori condition for the feasibility of (8).

V. SIMULATION RESULTS

We demonstrate the effectiveness of the proposed algorithm with two simulation examples.

We first consider a unicycle vehicle whose state consists of its position and angle $x = [p_1, p_2, \vartheta]^\top \in \mathbb{R}^3$, with $x(0) = [0, -0.05, 0.45]^\top$, and evolves according to the first-order dynamics $\dot{p}_1 = \cos(\vartheta)u_1$, $\dot{p}_2 = \sin(\vartheta)u_1$, $\dot{\vartheta} = u_2$; $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}$ are the vehicle's linear and angular velocities, representing the control inputs, and we choose $y = x$. The control inputs are constrained to evolve in the set $U = [-7,7]^2$. The goal is to track the planar figure-8 trajectory $y_\text{d} = [p_{\text{d},1}, p_{\text{d},2}, \vartheta_\text{d}]$, depicted in Fig. 1(b), with a period of 12 seconds. Note that the traditional funnel-base schemes (e.g. [1]–[7]), cannot guarantee tracking of $e = y - y_\text{d} \in \mathbb{R}^3$ for such a system. We choose the exponential funnel functions $\rho_1(t) = \rho_2(t) = \rho_3(t) = 0.95\exp(-t) + 0.05$ and $\alpha(*) = 0.01*$. Since there are no uncertainties, $\mathcal{C}_{\hat{\theta},\hat{f}}(t)$ is reduced to $\mathcal{C}(t) = \{x \in \mathbb{R}^n : \frac{1}{2}(1 - \|\xi\|^2) \geq 0\}$. We apply (10) via the QP (8) with $u_\text{n} = 0$ and design a 3rd-order polynomial for $\phi(\cdot)$ with $\delta = 0.1$ and $\delta_2 = 0.3$. In order to simulate a more realistic scenario, we apply a time-triggered version of (10a), with sampling frequency 200Hz; that is, the controller changes every 0.005 seconds. The results are depicted in Fig. 1 for $t = 15$ seconds; Fig. 1(a) shows the evolution of $e(t)$ along with the performance function $\rho(t)$, Fig. 1(b) shows the trajectories $y(t)$ and $y_\text{d}(t)$, and Fig. 1(e) depicts the control inputs $u(t)$. Although there exist singularities when $\nabla h(x)^\top g(x,t) = \|[\cos(\vartheta), \sin(\vartheta), 1]e\|^2 = 0$, the algorithm avoids such configurations and accomplishes the funnel-control objective. Moreover, note that the successful solution of (8) implies that $K_{u,\alpha}(x(t),t)$ is non-empty along the trajectory $x(t)$ of the closed-loop system. Finally, the oscillatory behaviour of $u(t)$ can be attributed to the optimization-based nature of (8). That is, the QP computes large control inputs to satisfy the inequality constraint only when the respective ξ_i, $i \in \{1,2,3\}$ are close to 1 and -1.

Secondly, we consider a simplified example of the cruise control problem [9]. We consider an autonomous vehicle, characterized by its longitudinal position $x_p \in \mathbb{R}$ that evolves according to the 2nd-order dynamics $\ddot{x}_p = F(x_p,t) + \frac{1}{m}u$, with $x_p(0) = 0$, $\dot{x}_p(0) = 2$, where $m = \frac{1}{\theta^*} = 0.7$ is the vehicle's mass and $F(x,t) = -\frac{1}{m}(0.1 - 2.5\dot{x}_p - 0.25\dot{x}_p^2 + 0.5\sin(t - \frac{\pi}{3}))$. The vehicle aims to follow a leader that evolves according to $\dot{x}_0 = 2t$, $x_0(0) = 5$, for $t \in [0,20]$ seconds, by tracking the reference trajectory $x_\text{d}(t) = x_0(t) - 2 + \frac{1}{2}\sin(\frac{1}{2}t)$. We set $y = \dot{x}_p + 0.1x_p$, which has to track $y_\text{d}(t) = \dot{x}_\text{d}(t) + 0.1x_\text{d}(t)$ within the funnel $\rho(t) = 0.9\exp(-t) + 0.1$. By setting $x_1 = x_p$, $x_2 = \dot{x}_p$, $x = [x_1, x_2]^\top$, and using (3), we conclude that $\bar{f} = 15$. We assume $F = 15$, $\Pi_\theta = (0.95, 1.5)$, and $\Pi_{c,\theta}$, $\Pi_{c,f}$ are defined by $p_\theta(q) = 0.3322q^2 - 74.7508$ and $p_f(q) = 80(q - 1.4286)^2 - 18$, respectively. We further use $\hat{f}(0) = 10$, $\hat{\theta}(0) = 1$, $k_f = 100$,

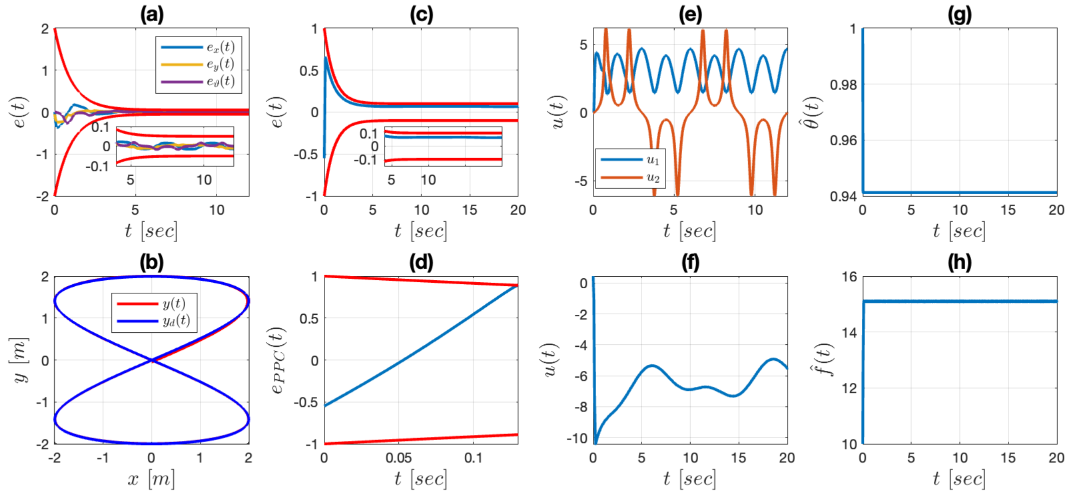

Fig. 1. Simulation results: (a), (c): The evolution of $\boldsymbol{e}(\boldsymbol{t})$, along with the funnel functions $\boldsymbol{\rho}(\boldsymbol{t})$ for the unicycle and cruise-control example, respectively; (b): The unicycle and reference trajectory, $\boldsymbol{y}(\boldsymbol{t})$ and $\boldsymbol{y}_{\mathsf{d}}(\boldsymbol{t})$, respectively; (d): The evolution of $\boldsymbol{e}(\boldsymbol{t})$, along with the funnel function $\boldsymbol{\rho}(\boldsymbol{t})$, for the cruise-control example driven by the controller of [2]; (e), (f): The evolution of $\boldsymbol{u}(\boldsymbol{t})$ for the unicycle and cruise-control example, respectively; (g), (h): The evolution of the adaptation variables $\hat{\boldsymbol{\theta}}(\boldsymbol{t})$ and $\hat{\boldsymbol{f}}(\boldsymbol{t})$m respectively, for the cruise-control example.

$k_\theta = 0.5$, $\alpha(*) = 0.01*$, and $U = [-50, 50]$. We apply a time-triggered version of (10), with frequency 500Hz, with a 3rd-order polynomial for $\phi(\cdot)$ with $\delta = 0.5$ and $\delta_2 = 0.8$. By noticing that, in this scalar scenario, the right-hand side of the constraint of (8) is equivalent to $-\|\nabla h(x)\|\hat{f}\|\xi\| - \xi^\top(\nabla h(x)^\top g(x,t,\hat{\theta})u - \dot{y}_{\mathsf{d}}(t) - \dot{\rho}(t)\xi) \geq -\rho(t)\alpha(b(x,t) - \frac{1}{2k_\theta}B_\theta^2 - \frac{1}{2k_f}B_f^2)$ and using the closed-form solution of (8), we conclude that $\sup_{(x(t),t,\hat{\theta},\hat{f})\in\mathbb{D}^\delta} \|u^*(x,t,\hat{\theta},\hat{f})\| \leq 40$, which complies with the input constraints. We further consider the PPC methodology of [2], where the control input is set as $u_{\text{PPC}} = -\log\left(\frac{1+\xi}{1-\xi}\right)$. The simulation results are depicted in Fig. 1 for $t = 20$ seconds; Fig. 1(c) shows the evolution of $e(t)$ along with the performance function $\rho(t)$, Fig. 1(f) shows $u(t)$, and Figs. 1(g), and 1(g)(h) depict the adaptation variables $\hat{\theta}(t)$ and $\hat{f}$, respectively. Finally, Fig. 1(d) depicts the evolution of the error $e_{PPC}(t)$ produced by u_{PPC}. One can conclude that the proposed algorithm guarantees the desired funnel specification, while the original PPC methodology fails to retain $e(t) \in (-\rho(t), \rho(t))$, which is attributed to the time-sampled control application.

VI. CONCLUSIONS

We present an algorithm that guarantees funnel-control specifications for a system with uncertain nonlinear dynamics by combining adaptive–control techniques with zeroing control barrier functions. Future work will address systems with higher output-relative degree and unknown $g(\cdot)$.

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