Asymptotic Consensus of Unknown Nonlinear Multi-Agent Systems with Prescribed Transient Response

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***Abstract*— We consider the asymptotic consensus problem for 2nd-order nonlinear multi-agent systems subject to predefined constraints for the system response, such as maximum overshoot or minimum convergence rate. We design a distributed discontinuous adaptive control protocol that guarantees that the inter-agent consensus errors evolve in a prescribed funnel while at the same time converging to zero. The multi-agent dynamics contain parametric and structural uncertainties, without boundedness or approximation/parametric factorization assumptions. The response of the closed loop multi-agent system is solely determined by the predefined funnel and is independent from the control gain selection. Finally, simulation results verify the theoretical findings.**

I. INTRODUCTION

Distributed control of networked multi-agent systems is an emerging and significant topic that has received a large amount of attention during the last decades due to the variety of its applications [1]–[3]. In such schemes, each agent calculates its own control signal based solely on local information in order to achieve a collaborative task with the other agents. A particular scheme of multi-agent systems consists of the leader-follower architecture, where a team of agents aims to follow a designated leader that holds information about the execution of a potential task [4], [5]. Leader-follower architectures find several applications in autonomous vehicle, systems biology, and power systems.

A significant challenge that arises is control of multi-agent systems with uncertain dynamics. Such uncertainties affect many practical engineering systems and consist of unknown dynamic parameters, modelling errors, or environmental disturbances. A large variety of works develops learning-based and adaptive control algorithms to compensate for the aforementioned uncertainties [6]–[9]. However, these works base their results on several limiting assumptions, such as global Lipschitz or boundedness conditions [5], [10], linearly parametrized dynamics with respect to unknown but constant parameters [11], and availability of a priori information on the dynamics [12].

Another important aspect of multi-agent systems is compliance with pre-defined transient and steady-state specifications. Such specifications encode properties with respect to overshoot, rate of convergence, and steady-state error of the trajectory of a system, providing thus guarantees related to safety and stabilization or tracking in a pre-defined time interval. A large class of works establishing pre-defined transient and steady-state specifications consist of funnel control algorithms, also known as prescribed-performance control [4], [13]–[15]. Such algorithms guarantee that the disagreement terms of the multi-agent system evolve in user-defined functions of time that form a time-varying funnel, which encodes the transient and steady-state specifications. Impressively, funnel-control algorithms are able to implicitly compensate large degrees of uncertainty in the dynamics. The cost of doing so, however, is the lack of asymptotic stability guarantees, i.e., the multi-agent disagreement terms remain in the funnel without necessarily converging to zero. Traditional funnel-based works achieve such a property only through funnels that "shrink" asymptotically, i.e., funnels that become arbitrarily narrow around zero as time grows to infinity. This, however, might yield undesired large inputs due to the small funnel values, and can be problematic in real-time systems. On the other hand, by achieving asymptotic stability, the funnels do not need to converge close to zero and can be used only to encode transient constraints for the system. Asymptotic stability with transient constraints for multi-agent was achieved in the works [15]–[17]; unlike the setup we consider in this paper, however, [15] considers the simplistic case of scalar states and constant control-input coefficients and [16], [17] do not consider dynamic uncertainties. Furthermore, asymptotic tracking subject to funnel transient constraints has been achieved in [18], [19] for single-agent systems assuming linear dynamics or parametric uncertainties.

In this paper, we consider the multi-agent synchronization problem subject to transient constraints imposed by a set of predefined funnels. The multi-agent system evolves to 2nd-order dynamics with unknown nonlinear terms. Our main contribution lies in the design of a distributed adaptive control algorithm that guarantees, not only confinement of the multi-agent disagreement terms in the given funnels, but also asymptotic stability from all initial conditions that satisfy the initial funnel constraints. The proposed algorithm is of low complexity, does not incorporate any information on the system model, and does not require global state boundedness assumptions or growth conditions. The transient performance of the multi-agent system is independent of the control gains and the system dynamics and depends solely on the pre-defined funnel.

The rest of the paper is structured as follows. Section II introduces preliminary background and notation. Section III describes the tackled problem and Section IV provides the proposed control protocol and the stability analysis. Finally, simulation results are given in Section V and Section VI concludes the paper.

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II. Notation and Preliminaries

A. Notation

The sets of real, positive real, and non-negative real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$, respectively; $\|x\|$ denotes the 2-norm of a vector $x \in \mathbb{R}^n$; The open and closed balls with respect to the 2-norm and with radius δ, centered at $x \in \mathbb{R}^n$, are denoted by $\mathcal{B}(x, \delta)$ and $\bar{\mathcal{B}}(x, \delta)$, respectively. Finally, $I_n \in \mathbb{R}^{n \times n}$ and $\otimes$ denote the identity matrix and Kronecker product, respectively.

B. Nonsmooth Analysis

Consider the following differential equation with a discontinuous right-hand side:

$$\dot{x} = f(x, t), \tag{1}$$

where $f : \mathcal{D} \times [t_0, \infty) \to \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$, is Lebesgue measurable and locally essentially bounded, uniformly in t. The Filippov regularization of f is defined as (see [20])

$$\mathsf{K}[f](x, t) := \bigcap_{\delta > 0} \bigcap_{\mu(\bar{N}) = 0} \overline{\text{co}}(f(\mathcal{B}(x, \delta) \backslash \bar{N}), t), \tag{2}$$

where $\bigcap_{\mu(\bar{N})=0}$ is the intersection over all sets $\bar{N}$ of Lebesgue measure zero, and $\overline{\text{co}}(E)$ is the convex closure of the set E.

Definition 1 (Def. 1 of [21]): A function $x : [t_0, t_1) \to \mathbb{R}^n$, with $t_1 > t_0$, is called a Filippov solution of (1) on $[t_0, t_1)$ if $x(t)$ is absolutely continuous and if, for almost all $t \in [t_0, t_1)$, it satisfies $\dot{x} \in \mathsf{K}[f](x, t)$, where $\mathsf{K}[f](x, t)$ is the Filippov regularization of $f(x, t)$.

Lemma 1 (Lemma 1 of [21]): Let $x(t)$ be a Filippov solution of (1) and $V : \mathcal{D} \times [t_0, t_1) \to \mathbb{R}$ be a locally Lipschitz, regular function. Then, $V(x(t), t)$ is absolutely continuous, $\dot{V}(x(t), t) = \frac{\partial}{\partial t} V(x(t), t)$ exists almost everywhere (a.e.), i.e., for almost all $t \in [t_0, t_1)$, and $\dot{V}(x(t), t) \overset{\text{a.e}}{\in} \dot{\tilde{V}}(x(t), t)$, where

$$\dot{\tilde{V}} := \bigcap_{\xi \in \partial V(x,t)} \xi^\top \begin{bmatrix} \mathsf{K}[f](x, t) \\ 1 \end{bmatrix},$$

and $\partial V(x, t)$ is Clarke's generalized gradient at (x, t) [21].

Theorem 1 (Corollary 2 of [21]): For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be an open and connected set containing 0 and suppose that f is Lebesgue measurable and $x \mapsto f(x, t)$ is essentially locally bounded, uniformly in t. Let $V : \mathcal{D} \times [t_0, t_1) \to \mathbb{R}$ be locally Lipschitz and regular such that $W_1(x) \leq V(x, t) \leq W_2(x)$, $\forall t \in [t_0, t_1)$, $x \in \mathcal{D}$, and

$$z \leq -W(x(t)), \quad \forall z \in \dot{\tilde{V}}(x(t), t), \ t \in [t_0, t_1), \ x \in \mathcal{D}, \tag{3}$$

where W_1 and W_2 are continuous positive definite functions and W is a continuous positive semi-definite on $\mathcal{D}$. Choose $r > 0$ and $c > 0$ such that $\bar{\mathcal{B}}(0, r) \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$. Then, for all Filippov solutions $x : [t_0, t_1) \to \mathbb{R}^n$ of (1), with $x(t_0) \in \mathbb{D} := \{x \in \bar{\mathcal{B}}(0, r) : W_2(x) \leq c\}$, it holds that $t_1 = \infty$, $x(t) \in \mathbb{D}$, $\forall t \in [t_0, \infty)$, and $\lim_{t \to \infty} W(x(t)) = 0$.

We now introduce the next theorem that relaxes the inequality condition (3).

Theorem 2: For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be an open and connected set containing 0 and suppose that f is Lebesgue measurable and $x \mapsto f(x, t)$ is essentially locally bounded, uniformly in t. Let $V : \mathcal{D} \times [t_0, t_1) \to \mathbb{R}$ be locally Lipschitz and regular such that $W_1(x) \leq V(x, t) \leq W_2(x)$, $\forall t \in [t_0, t_1)$, $x \in \mathcal{D}$, and

$$\begin{aligned} W_1(x) \leq& V(x, t) \leq W_2(x), \ \forall t \in [t_0, t_1), x \in \mathcal{D}, & \text{(4a)} \\ z \leq& -W(x) + \alpha \exp(-\beta t), & \\ & \forall z \in \dot{\tilde{V}}(x, t), \ t \in [t_0, t_1), x \in \mathcal{D}, & \text{(4b)} \end{aligned}$$

where W_1 and W_2 are continuous positive definite functions, W is a continuous positive semi-definite on $\mathcal{D}$, and α, β are positive constants. Choose $r > 0$ and $c > 0$ such that $\bar{\mathcal{B}}(0, r) \subset \mathcal{D}$ and $c + \frac{\alpha}{\beta} \exp(-\beta t_0) < \min_{\|x\|=r} W_1(x)$. Then, for all Filippov solutions $x : [t_0, t_1) \to \mathbb{R}^n$ of (1), with $x(t_0) \in \mathbb{D} := \{x \in \bar{\mathcal{B}}(0, r) : W_2(x) \leq c\}$, it holds that $t_1 = \infty$, $x(t) \in \mathbb{D}$, for all $t \in [t_0, \infty)$, and $\lim_{t \to \infty} W(x(t)) = 0$.

Proof: Firstly, inequality (4b) and Lemma 1 suggest that $\dot{V}(x(t), t) \overset{\text{a.e}}{\leq} -W(x(t)) + \alpha \exp(-\beta t)$ for any arbitrary Filippov solution of (1) and all $t \geq t_0$. We argue that the latter condition implies that

$$V(x(t), t) \leq V(x(t_0), t_0) + \frac{\alpha}{\beta} \big(\exp(-\beta t_0) - \exp(-\beta t) \big), \tag{5}$$

for all $t \geq t_0$. Indeed, aiming to reach a contradiction, let a $t > t_0$ such that $V(x(t), t) > V(x(t_0), t_0) + \frac{\alpha}{\beta} \big(\exp(-\beta t_0) - \exp(-\beta t) \big)$. Then, it holds that

$$\begin{aligned} \int_{t_0}^{t} \dot{V}(x(\sigma), \sigma) d\sigma &= V(x(t), t) - V(x(t_0), t_0) \\ &> \frac{\alpha}{\beta} \big(\exp(-\beta t_0) - \exp(-\beta t) \big). \end{aligned}$$

It follows that $\dot{V}(x(t), t) > \alpha \exp(-\beta t)$ on a set of positive measure, which contradicts $\dot{V}(x(t), t) \leq \alpha \exp(-\beta t)$, a.e.

Define now the sets

$$\begin{aligned} \mathcal{S}_{1,\ell} &:= \{x \in \mathcal{B}(0, r) : W_1(x) \leq \ell\} \\ \mathcal{S}_{2,\ell} &:= \{x \in \mathcal{B}(0, r) : W_2(x) \leq \ell\} \\ \Omega_\ell(t) &:= \{x \in \mathcal{B}(0, r) : V(x, t) \leq \ell\}, \end{aligned}$$

for a positive constant ℓ, as well as the constant $\kappa := c + \frac{\alpha}{\beta} \exp(-\beta t_0)$. Note that, due to the fact that $\kappa < \min_{\|x\|=r} W_1(x)$ and (4a), it holds that

$$\mathcal{S}_{2,\ell} \subset \Omega_\ell(t) \subset \mathcal{S}_{1,\ell} \subset \mathcal{B}(0, r) \subset \mathcal{D},$$

for $\ell \in \{c, \kappa\}$ and all $t \geq t_0$. Moreover, (5) implies that $V(x(t), t) \leq V(x(t_0), t_0) + \frac{\alpha}{\beta} \exp(-\beta t_0)$. Therefore, for any $t_0 \geq 0$ and any $x(t_0) \in \Omega_c(t_0)$, the solution starting at $(x(t_0), t_0)$ stays in $\Omega_\kappa(t)$, for every t_0. Hence, since $\mathcal{S}_{2,c} \subset \Omega_c(t_0)$, any solution starting in $\mathcal{S}_{2,c}$ stays in $\Omega_\kappa(t)$ for all future times. Finally, since $\Omega_\kappa(t) \subset \mathcal{S}_{1,\kappa} \subset \mathcal{B}(0, r)$, it holds that $\|x(t)\| < r$, for all $t \geq t_0$.

What is left to prove is that $\lim_{t\to\infty} W(x(t)) = 0$. Towards that end, we define first $\widetilde{W} : \mathcal{D} \times [t_0, \infty)$, with

$$\widetilde{W}(x,t) := W(x) - \alpha \exp(-\beta t).$$

Consequently, one obtains from (4b) and Lemma 1 that

$$\int_{t_0}^{t} \widetilde{W}(x(\tau))d\tau \leq -\int_{t_0}^{t} \dot{V}(x(\tau),\tau)d\tau = V(x(t_0),t_0) - V(x(t),t) \leq V(x(t_0),t_0).$$

Therefore, $\int_{t_0}^{t} \widetilde{W}(x(\tau))d\tau$ is bounded for all $t \geq t_0$. Moreover, $\lim_{t\to\infty} \int_{t_0}^{t} \widetilde{W}(x(\tau))d\tau$ is guaranteed to exist since (i) $\lim_{t\to\infty} \int_{t_0}^{t} \alpha \exp(-\beta t) = \lim_{t\to\infty} \frac{\alpha}{\beta}(\exp(-\beta t_0) - \exp(-\beta t)) = \frac{\alpha}{\beta}\exp(-\beta t_0)$, and (ii) $\int_{t_0}^{t} W(x(\tau))d\tau$ is monotonically non-decreasing due to the positive semi-definiteness of $W(x)$. Since $x(t)$ is locally absolutely continuous and f is essentially locally bounded, uniformly in t, $x(t)$ is uniformly continuous. Because $\widetilde{W}(x,t)$ is continuously differentiable in t and continuous in x, and $x \in \mathcal{B}(0,r) \subset \bar{\mathcal{B}}(0,r)$, $W(x(t),t)$ is uniformly continuous in t on $[t_0,\infty)$. Therefore, by using Barbalat's lemma [22], we conclude that $\lim_{t\to\infty} \widetilde{W}(x(t),t) = \lim_{t\to\infty} \big(W(x(t)) - \alpha\exp(-\beta t)\big) = 0$, leading to $\lim_{t\to\infty} W(x(t)) = 0$.

■

III. PROBLEM FORMULATION

Consider a MIMO multi-agent team comprised of a leader and N followers, with the leading agent acting as an exosystem that generates a desired command/reference trajectory for the multi-agent team. The followers evolve according to the 2nd-order dynamics

$$\dot{x}_{i,1} = x_{i,2} \tag{6a}$$
$$\dot{x}_{i,2} = f_i(x_i,\zeta_i,t) + g_i(x_i,\zeta_i)u_i \tag{6b}$$
$$\dot{\zeta}_i = f_{\zeta_i}(x_i,\zeta_i,t) \tag{6c}$$

for $i \in \mathcal{N} := \{1,\ldots,N\}$, where $\zeta_i \in \mathbb{R}^{n_\zeta}$, $x_i := [x_{i,1}^\top, x_{i,2}^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^n$ are the states of agent $i \in \mathcal{N}$, with $n_\zeta \geq 2$ and $n \geq 2$, which are available for measurement, $f_i : \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $g_i : \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ are *unknown* vector fields, and $u_i \in \mathbb{R}^n$ is the control input of agent $i \in \mathcal{N}$. We make the following regularity assumptions for $f_i(\cdot)$ and $g_i(\cdot)$:

Assumption 1: The maps $(x_i,\zeta_i) \mapsto f_i(x_i,\zeta_i,t) : \mathbb{R}^{2n+n_\zeta} \to \mathbb{R}^n$ and $(x_i,\zeta_i) \mapsto g_i(x_i,\zeta_i,t) : \mathbb{R}^{2n+n_\zeta} \to \mathbb{R}^{n\times n}$ are locally Lipschitz for each $t \in \mathbb{R}_{\geq 0}$ and the maps $t \mapsto f_i(x_i,\zeta_i,t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are uniformly bounded for each $(x_i,\zeta_i) \in \mathbb{R}^n \times \mathbb{R}^{n_\zeta}$, for all $i \in \mathcal{N}$.

Assumption 2: The matrices

$$\widetilde{g}_i(x_i,\zeta_i,t) := g_i(x_i,\zeta_i,t) + g_i(x_i,\zeta_i,t)^\top,\ i \in \mathcal{N},$$

are positive definite, for all $(x_i,\zeta_i,t) \in \mathbb{R}^{2n+n_\zeta} \times \mathbb{R}_{\geq 0}$.

Assumption 3: There exist sufficiently smooth functions $U_{\zeta_i} : \mathbb{R}^{n_\zeta} \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\gamma}_{\zeta_i}(\cdot)$, $\bar{\gamma}_{\zeta_i}(\cdot)$, $\gamma_{\zeta_i}(\cdot)$ such that $\underline{\gamma}_{\zeta_i}(\|\zeta_i\|) \leq U_{\zeta_i}(\zeta_i) \leq \bar{\gamma}_{\zeta_i}(\|\zeta_i\|)$, and

$$\left(\frac{\partial U_{\zeta_i}}{\partial \zeta_i}\right)^\top f_{\zeta_i}(x_i,\zeta_i,t) \leq -\gamma_{\zeta_i}(\|\zeta_i\|) + \pi_{\zeta_i}(x_i,\zeta_i,t),$$

for all $i \in \mathcal{N}$, where $x_i \mapsto \pi_{\zeta_i}(x_i,\zeta_i,t) : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ is continuous and class $\mathcal{K}_\infty$ for each $(\zeta_i,t) \in \mathbb{R}^{n_\zeta} \times \mathbb{R}_{\geq 0}$, and $(\zeta_i,t) \mapsto \pi_{\zeta_i}(x_i,\zeta_i,t) : \mathbb{R}^{n_\zeta} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly bounded for each $x \in \mathbb{R}^{2n}$, for all $i \in \mathcal{N}$.

Assumption 2 is a sufficient controllability condition (similar to the ones considered in a variety of works, e.g., [4], [5], [12]), and Assumption 3 suggests that ζ_i are input-to-state practically stable implying stable zero (internal) dynamics.

We write the multi-agent dynamics (6) in vector form

$$\dot{x}_1 = x_2 \tag{7a}$$
$$\dot{x}_2 = f(x,\zeta,t) + g(x,\zeta,t)u \tag{7b}$$
$$\dot{\zeta} = f_\zeta(x,\zeta,t) \tag{7c}$$

where we use the stacked-vector notation

$$\begin{aligned}
x_1 &:= [x_{1,1}^\top,\ldots,x_{N,1}]^\top \in \mathbb{R}^{Nn}\\
x_2 &:= [x_{1,2}^\top,\ldots,x_{N,2}]^\top \in \mathbb{R}^{Nn}\\
x &:= [x_1^\top, x_2^\top]^\top \in \mathbb{R}^{2Nn}\\
\zeta_i &:= [\zeta_1^\top,\ldots,\zeta_N^\top]^\top \in \mathbb{R}^{Nn_\zeta}\\
u &:= [u_1^\top,\ldots,u_N^\top]^\top \in \mathbb{R}^{Nn}\\
f &:= [f_1^\top,\ldots,f_N^\top]^\top \in \mathbb{R}^{Nn}\\
f_\zeta &:= [f_{\zeta_1}^\top,\ldots,f_{\zeta_N}^\top]^\top \in \mathbb{R}^{Nn_\zeta}\\
g &:= \mathrm{diag}\{g_1,\ldots,g_N\} \in \mathbb{R}^{Nn\times Nn}
\end{aligned}$$

We use an undirected graph $\mathcal{G} := (\mathcal{N},\mathcal{E})$ to model the communication among the agents, with $\mathcal{N}$ being the index set of the agents, and $\mathcal{E} \subseteq \mathcal{N}\times\mathcal{N}$ being the respective edge set, with $(i,i) \notin \mathcal{E}$ (i.e., simple graph). The adjacency matrix associated with the graph $\mathcal{G}$ is denoted by $\mathcal{A} := [a_{ij}] \in \mathbb{R}^{N\times N}$, with $a_{ij} \in \{0,1\}$, $i,j \in \{1,\ldots,N\}$. If $a_{ij} = 1$, then agent i obtains information regarding the state x_j of agent j (i.e., $(i,j) \in \mathcal{E}$), whereas if $a_{ij} = 0$ then there is no state-information flow from agent j to agent i (i.e., $(i,j) \notin \mathcal{E}$). Furthermore, the set of neighbors of agent i is denoted by $\mathcal{N}_i := \{j \in \mathcal{N} : (i,j) \in \mathcal{E}\}$, and the degree matrix is defined as $\mathcal{D} := \mathrm{diag}\{|\mathcal{N}_1|,\ldots,|\mathcal{N}_N|\}$. Since the graph is undirected, the adjacency is a mutual relation, i.e., $a_{ij} = a_{ji}$, rendering $\mathcal{A}$ symmetric. The *Laplacian* matrix of the graph is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$ and is also symmetric. The graph is *connected* if there exists a path between any two agents. For a connected graph, it holds that $\mathcal{L}\bar{1} = 0$, where $\bar{1}$ is the vector of ones of appropriate dimension.

Regarding the leader agent, we denote its state variables by $x_0 := [x_{0,1}^\top, x_{0,2}^\top]^\top \in \mathbb{R}^{2n}$, evolving according to $\dot{x}_{0,1} = x_{0,2} = q_0$, where q_0 is a positive constant, such that $x_{0,1}(t)$ is bounded for all finite t. However, the leader provides its state only to a subgroup of the N agents. In particular, we model the access of the follower agents to the leader's state via a diagonal matrix $\mathcal{B} := \mathrm{diag}\{b_1,\ldots,b_N\} \in \mathbb{R}^{N\times N}$; if $b_i = 1$, then the ith agent has access to the leader's state, whereas it does not if $b_i = 0$, for $i \in \mathcal{N}$. We further define $H := (\mathcal{L}+\mathcal{B}) \otimes I_n$.

The control objective is the design of a distributed control algorithm for the followers, using relative state feedback, that asymptotically stabilizes the disagreement vectors

$$\delta_i(t) \coloneqq x_{i,1}(t) - x_{0,1}(t) \tag{8}$$

to zero, for all $i \in \mathcal{N}$. At the same time, as discussed in Section I, we aim at imposing a certain predefined behaviour for the transient response of the multi-agent system. Note, however, that $\delta_i(t)$ in (8) is not accessible by the agents that are not connected to the leader, i.e., agents for which $b_i = 0$. Therefore, we formulate the transient behaviour on the error variables

$$\begin{aligned} e_i \coloneqq& [e_{i,1}, \dots, e_{i,n}]^\top \\ \coloneqq& \sum_{j\in\mathcal{N}_i} a_{ij}(x_{i,1} - x_{j,1}) + b_i(x_{i,1} - x_{0,1}) \end{aligned} \tag{9}$$

for $i \in \mathcal{N}$, which will define the subsequent control design. The transient behaviour we aim at imposing is inspired by funnel-control techniques [23]–[25]. More specifically, given n predefined funnels for each agent, described by the exponentially decaying functions (also called performance functions [24]) $\rho_{i,k} : \mathbb{R}_{\geq 0} \to [\underline{\rho}_{i,k}, \bar{\rho}_{i,k}] \subset \mathbb{R}_{>0}$, with $\rho_{i,k}(t) \coloneqq (\bar{\rho}_{i,k} - \underline{\rho}_{i,k})\exp(-l_{i,k}t) + \underline{\rho}_{i,k}$, we aim at guaranteeing that[1] $-\rho_{i,k}(t) < e_{i,k}(t) < \rho_{i,k}(t)$, for all $t \geq 0$, $i \in \mathcal{N}$, and $k \in \mathcal{K} \coloneqq \{1, \dots, n\}$, given that $-\rho_{i,k}(0) < e_{i,k}(0) < \rho_{i,k}(0)$, for all $i \in \mathcal{N}, k \in \mathcal{K}$. These functions can encode maximum overshoot or convergence rate properties. Note that, compared to the majority of the related works on multi-agent funnel control (e.g., [4], [5]), we do not require arbitrarily small final values $\underline{\rho}_{i,k} = \lim_{t\to\infty}\rho_{i,k}(t)$, which would achieve convergence of $e_{i,k}(t)$ arbitrarily close to zero, since one of the objectives is actual *asymptotic stability*. Formally, the problem statement is the following:

Problem 1: Consider the multi-agent system (6) and n funnels for each agent described by the functions $\rho_{i,k}$. Design a distributed control algorithm u such that $\lim_{t\to\infty} e_i(t) = 0$ and $-\rho_{i,k}(t) < e_{i,k}(t) < \rho_{i,k}(t)$, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$, $t \geq 0$, and all closed loop signals remain bounded.

To solve the aforementioned problem, we need the following assumption on the graph connectivity:

Assumption 4: The graph $\mathcal{G}$ is connected and there exists at least one $i \in \mathcal{N}$ such that $b_i = 1$.
Assumption 4 implies that $H = (\mathcal{L}+\mathcal{B})\otimes I_n$ is an irreducibly diagonally dominant M-matrix [26]. An M-matrix is a square matrix having its off-diagonal entries non-positive and all principal minors nonnegative; thus H is positive definite.

By stacking all e_i and using (9) and (8), one obtains

$$e \coloneqq [e_1^\top, \dots, e_N^\top]^\top = H\delta \tag{10}$$

where $\delta \coloneqq [\delta_1^\top, \dots, \delta_N^\top]^\top$. Therefore, since $H = (\mathcal{L}+\mathcal{B}) \otimes I_n$ and $\mathcal{L}+\mathcal{B}$ is positive definite owing to Assumption 4, we conclude that

$$\|\delta\| \leq \frac{\|e\|}{\lambda_{\min}(\mathcal{L}+\mathcal{B})}. \tag{11}$$

[1] The results can be extended to non-symmetric funnels.

Therefore, the control-design specification $\lim_{t\to\infty} e_i(t) = 0$ leads to $\lim_{t\to\infty} \delta_i(t) = 0$, $i \in \mathcal{N}$. Similarly, the transient-state specifications imposed to $e_{i,k}(t)$ via the funnels $\rho_{i,k}(t)$ can be directly translated into respective specifications for $\delta_i(t)$. Although $\lambda_{\min}(\mathcal{L}+\mathcal{B})$ is related to the global topology of the network and is not known by the agents, one can employ the lower bound [27] $\lambda_{\min}(\mathcal{L}+\mathcal{B}) \geq \Pi(N) \coloneqq \frac{\frac{N-1}{2}^{\frac{N-1}{2}}}{N^2+N-1}$, which depends only the number of agents, to derive

$$\|\delta\| \leq \frac{\|e\|}{\Pi(N)}. \tag{12}$$

which translates the transient-state specification of $e(t)$ to transient-state specifications for $\delta(t)$.

IV. MAIN RESULTS

The proposed solution to Problem 1 is based on the error transformation proposed in [24], which converts the constrained error behaviour $-\rho_{i,k}(t) < e_{i,k}(t) < \rho_{i,k}(t)$ to an unconstrained one, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$. More specifically, we define the error transformations $\varepsilon_{i,k} \in \mathbb{R}$ according to:

$$e_{i,k} = \rho_{i,k}T(\varepsilon_{i,k}), \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{13}$$

where $T : \mathbb{R} \to (-1, 1)$ is a smooth, strictly increasing function, with $T(0) = 0$. Since T is increasing, the inverse mapping $T^{-1} : (-1, 1) \to \mathbb{R}$ is well-defined, and

$$\lim_{z\to-\infty} T(z) = -1 \quad \lim_{z\to+\infty} T(z) = 1, \tag{14}$$

and hence, if $\varepsilon_{i,k}$ remains bounded in a compact set, the desired funnel objective $-\rho_{i,j}(t) < e_{i,k}(t) < \rho_{i,k}(t)$ is achieved, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$. We further require that

$$|z| \leq \left|\frac{\partial T^{-1}(z)}{\partial z}T^{-1}(z)\right|, \quad \forall z \in (-1, 1). \tag{15}$$

A possible choice that satisfies the aforementioned specifications is $T(z) \coloneqq \frac{\exp(z)-1}{\exp(z)+1}$.

From (13), we obtain $\varepsilon_{i,k} = T^{-1}\left(\frac{e_{i,k}}{\rho_{i,k}}\right)$, and by defining $\varepsilon_i \coloneqq [\varepsilon_{i,1}, \dots, \varepsilon_{i,n}]^\top$, we obtain

$$\begin{aligned} \dot{\varepsilon}_i =& r_i\rho_i^{-1}\bigg(\sum_{j\in\mathcal{N}_i} a_{ij}(x_{i,2} - x_{j,2}) \\ &+ b_i(x_{i,2} - x_{0,2}) - \dot{\rho}_i e_i \rho_i^{-1}\bigg), \end{aligned} \tag{16}$$

where $\rho_i \coloneqq \mathrm{diag}\{[\rho_{i,k}]_{k\in\mathcal{K}}\} \in \mathbb{R}^{n\times n}$, $r_i \coloneqq \mathrm{diag}\left\{\left[\frac{\partial T^{-1}(z)}{\partial z}\Big|_{z=\frac{e_{i,k}}{\rho_{i,k}}}\right]_{k\in\mathcal{K}}\right\}$, for $i \in \mathcal{N}$. By further defining $\varepsilon \coloneqq [\varepsilon_1^\top, \dots, \varepsilon_N^\top]^\top$ and using (16) and (10), we obtain

$$\dot{\varepsilon} = r\rho^{-1}\left(H(x_2 - \bar{x}_{0,2}) - \dot{\rho}\rho^{-1}e\right), \tag{17}$$

where $r \coloneqq \mathrm{diag}\{[r_i]_{i\in\mathcal{N}}\}$, $\rho \coloneqq \mathrm{diag}\{[\rho_i]_{i\in\mathcal{N}}\}$, and $\bar{x}_{0,2} \coloneqq [x_{0,2}^\top, \dots, x_{0,2}^\top]^\top \in \mathbb{R}^{Nn}$. Since the leader signal $x_{0,2}$ is unknown to the agents for which $b_i = 0$, we define the respective estimate variables $\hat{v}_i \in \mathbb{R}^n$, $i \in \mathcal{N}$ of an observer system. The proposed control algorithm follows a backstepping-like methodology: we define first the reference signal $x_{i,v}$ for each agent as

$$x_{i,v} := -\kappa_{i,1} r_i \rho_i^{-1} \varepsilon_i + \hat{v}_i, \quad i \in \mathcal{N} \tag{18}$$

where $\kappa_{i,1}$ is a positive constant, for all $i \in \mathcal{N}$. We further design the dynamics of $\hat{v}_i$ as

$$\dot{\hat{v}}_i = -\sum_{i \in \mathcal{N}_j} (\hat{v}_i - \hat{v}_j) - b_i(\hat{v}_i - x_{0,2}), \quad i \in \mathcal{N}. \tag{19}$$

Next, we define the associated errors

$$e_{v_i} := [e_{v_i,1}^\top, \dots, e_{v_i,n}^\top]^\top := x_{i,2} - x_{i,v} \tag{20}$$

Proceeding in a similar fashion, we define a funnel for each $e_{v_i,k}$ described by the functions $\rho_{v_i,k} : \mathbb{R}_{\geq 0} \to [\underline{\rho}_{v_i,k}, \bar{\rho}_{v_i,k}] \subset \mathbb{R}_{>0}$, where $\underline{\rho}_{v_i,k}, \bar{\rho}_{v_i,k}$ are the positive lower and upper bounds, respectively, with the constraint $\rho_{v_i,k}(0) > |e_{v_i,k}(0)|$, $i \in \mathcal{N}$, $k \in \mathcal{K}$. Note that agent i can calculate $e_{v_i,k}(0)$ since it is a function of its own and its neighbours' state and the funnel functions. Moreover, note that the functions $\rho_{v_i,k}$ represent an artificial funnel, in the sense that they are not part of the given specification (as $\rho_{i,k}$). The constraints they need to satisfy concern boundedness, positivity, and initial compliance with respect to the respective errors, i.e., $\rho_{v_i,k}(0) > |e_{v_i,k}(0)|$, $i \in \mathcal{N}$, $k \in \mathcal{K}$.

Next, we define the open set $\mathcal{D}_{u,t} := \{(x,t) \in \mathbb{R}^{2Nn} \times \mathbb{R}_{\geq 0} : \rho_i(t)^{-1} e_i \in (-1,1)^n, \rho_{v_i}(t)^{-1} e_{v_i} \in (-1,1)^n, i \in \mathcal{N}\}$, with $\rho_{v_i} := \text{diag}\{[\rho_{v_i,k}]_{k \in \mathcal{K}}\}$, $i \in \mathcal{N}$, and design the distributed control law $u_i : \mathcal{D}_{u,t} \to \mathbb{R}^n$ as

$$u_i = -\kappa_{i,2} \rho_{v_i}^{-1} r_{v_i} \varepsilon_{v_i} - \kappa_{i,3} \hat{d}_i \rho_{v_i}^{-1} s_{v_i} \tag{21}$$

where, for all $i \in \mathcal{N}$,

$$s_{v_i} := \begin{cases} \frac{r_{v_i}\varepsilon_{v_i}}{\|r_{v_i}\varepsilon_{v_i}\|}, & \text{if } \varepsilon_{v_i} \neq 0 \\ 0, & \text{otherwise} \end{cases},$$

$\varepsilon_{v_i} := [\varepsilon_{v_i,1}, \dots, \varepsilon_{v_i,n}]^\top$, $\varepsilon_{v_i,k} := T^{-1}\left(\frac{e_{v_i,k}}{\rho_{v_i,k}}\right)$, $k \in \mathcal{K}$, $r_{v_i} := \text{diag}\{[r_{v_1,k}]_{k \in \mathcal{K}}\}$, $r_{v_i,k} := \frac{\partial T^{-1}(z)}{\partial z}\Big|_{z = \frac{e_{v_i,k}}{\rho_{v_i,k}}}$, $k \in \mathcal{K}$, $\kappa_{i,2}, \kappa_{i,3}$ are positive gains, and $\hat{d}_i$ are adaptive variable gains, subject to the constraint $\hat{d}_i(0) \geq 0$, and dynamics

$$\dot{\hat{d}}_i = \gamma_i \|r_{v_i} \varepsilon_{v_i}\|, \tag{22}$$

where $\gamma_i > 0$ are positive constant gains, for all $i \in \mathcal{N}$.

Remark 1: The control design procedure follows closely our previous work on single-agent systems [28], inspired by the original prescribed performance methodology [4], [24], [29]. Traditionally, the desired signals and control law consist only of proportional terms with respect to the transformed errors ε_i, ε_{v_i}, $i \in \mathcal{N}$, which are guaranteed to be ultimately bounded. In this work, the incorporation of the extra terms in (21) achieves convergence of the transformed errors to zero, guaranteeing thus asymptotic stability.

Remark 2: Note that no information regarding the dynamic model is incorporated in the control protocol (18)-(22). Furthermore, no a-priori gain tuning is needed and, as the next theorem states, the solution of Problem 1 is guaranteed from *all* initial conditions that satisfy $-\rho_{i,k}(0) < e_{i,k}(0) < \rho_{i,k}(0)$, $i \in \mathcal{N}$. As will be revealed subsequently, the adaptive gains $\hat{d}_i$ compensate the unknown dynamic terms, which are proven to be bounded due to the confinement of the states in the prescribed funnels. Finally, note that each agent uses only local information from its neighbours to compute (18) and (21).

Theorem 3: Consider a system subject to the dynamics (6) and let funnels $\rho_{i,k}$ as described in Problem 1. Then. under Assumptions 1-4, the control algorithm (18)-(22) guarantees the solution of Problem 1 from all initial conditions that satisfy $-\rho_{i,k}(0) < e_{i,k}(0) < \rho_{i,k}(0)$, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$.

Proof: The intuition of the subsequent proof is as follows: We first show the existence of at least one Filippov solution of the closed-loop system in $\mathcal{D}_{u,t}$ for a time interval $\mathcal{I} \subseteq [t_0, \infty)$. Next, we prove that for any of these solutions, the state remains bounded in $\mathcal{I}$ by bounds independent of the endpoint of $\mathcal{I}$. Hence, the dynamic terms of (6) are also upper bounded by a term, which we aim to compensate via the adaptation gains $\hat{d}_i$, $i \in \mathcal{N}$.

We start by defining some terms that will be used in the subsequent analysis: $K_1 := \text{diag}\{\kappa_{i,1}, \dots, \kappa_{N,1}\} \otimes I_n$, $\bar{H} := \lambda_{\max}(H)$, $\underline{H}_1 := \lambda_{\min}(K_1 H K_1)$, $M_p := \max_{i \in \mathcal{N}, k \in \mathcal{K}}\{\bar{\rho}_{i,k}\}$, $m_p := \min_{i \in \mathcal{N}, k \in \mathcal{K}}\{\underline{\rho}_{i,k}\}$, $M_v := \max_{i \in \mathcal{N}, k \in \mathcal{K}}\{\bar{\rho}_{v_i,k}\}$, $m_v := \min_{i \in \mathcal{N}, k \in \mathcal{K}}\{\underline{\rho}_{v_i,k}\}$, $M_{\dot{v}} := \max_{i \in \mathcal{N}, k \in \mathcal{K}}\{\sup_{t \geq 0}\{|\dot{\rho}_{v_i,k}(t)|\}\}$, $\underline{\lambda}_i := \min_{i \in \mathcal{N}}\left(\rho_{v_i}(t)^{-1} \widetilde{g}_i(x_i, \zeta_i, t) \rho_{v_i}(t)^{-1}\right)$, $\beta_i := (k_{i,3}\underline{\lambda}_i)^{-1}$, $i \in \mathcal{N}$, $\underline{r} := \inf_{z \in (-1,1)} \frac{\partial T^{-1}(z)}{\partial z}$. Note that all the aforementioned terms are strictly positive. Moreover, in view of (14), it holds that $\arg\inf_{z \in (-1,1)} \frac{\partial T^{-1}(z)}{\partial z} \in (-1,1)$. By employing (21), (22), the closed loop system becomes

$$\dot{x}_1 = x_2, \tag{23a}$$
$$\dot{x}_2 \in f(x,\zeta) + g(x,\zeta,t)(\mathsf{K}[u](x,\hat{d},t)), \tag{23b}$$
$$\dot{\zeta} = f_\zeta(x,\zeta,t) \tag{23c}$$
$$\dot{\hat{d}} = f_d(x,t), \tag{23d}$$

where $\hat{d} := [\hat{d}_1, \dots, \hat{d}_N]^\top \in \mathbb{R}^N$, $f_d := [\gamma_1\|r_{v_1}\varepsilon_{v_1}\|, \dots, \gamma_N\|r_{v_N}\varepsilon_{v_N}\|] \in \mathbb{R}^N$, and $\mathsf{K}[u](x,d,t)$ is the Filippov regularization of $u(x,d,t)$, formed by substituting the terms s_{v_i} with their regularized terms, which are $\mathsf{S}_{v_i} = \frac{r_{v_i}\varepsilon_{v_i}}{\|r_{v_i}\varepsilon_{v_i}\|}$ if $\|r_{v_i}\varepsilon_{v_i}\| \neq 0$ and $\mathsf{S}_{v_i} \in (-1,1)^n$ otherwise, $i \in \mathcal{N}$. Note that it holds $(r_{v_i}\varepsilon_{v_i})^\top \mathsf{S}_{v_i} = \|r_{v_i}\varepsilon_{v_i}\|$, $i \in \mathcal{N}$. Define now $\widetilde{x} := [x^\top, \zeta^\top, \hat{d}^\top]^\top \in \mathbb{R}^{2Nn + Nn_\zeta + N}$ and consider the open set $\mathcal{D}_c := \{(\widetilde{x}, t) \in \mathbb{R}^{2Nn + Nn_\zeta + N} \times \mathbb{R}_{\geq 0} : (x,t) \in \mathcal{D}_{u,t}\}$. Since $\rho_{i,k}(0) > |e_{i,k}(0)|$ and $\rho_{v_i,k}(0) > |e_{v_i,k}(0)|$, for all $i \in \mathcal{N}, k \in \mathcal{K}$, the set $\mathcal{D}_c$ is nonempty. Moreover, the right hand-side of (23) is Lebesgue measurable and locally Lipschitz in $\widetilde{x}$ over the set $\{\widetilde{x} : (\widetilde{x}, t) \in \mathcal{D}_c\}$, and Lebesgue measurable in t over the set $\{t : (\widetilde{x}, t) \in \mathcal{D}_c\}$. Hence, according to Prop. 3 of [30], for each initial condition $(\widetilde{x}(0), 0) \in \mathcal{D}_c$, there exists at least one Filippov solution $\widetilde{x}(t)$ of (23), defined in $\mathcal{I} := [0, t_{\max})$, where $t_{\max} > 0$, such that $(\widetilde{x}(t), t) \in \mathcal{D}_c$, for all $t \in \mathcal{I}$. By applying the transformation $T(\cdot)^{-1}$, we conclude the existence of the respective Filippov solutions $\varepsilon_i(t), \varepsilon_{v_i}(t)$, $i \in \mathcal{N}$, for all $t \in \mathcal{I}$. Let now $\widetilde{x}(0)$ denote the initial condition of the system (23) satisfying $(\widetilde{x}(0), 0) \in \mathcal{D}_c$ and consider the family of Filippov solutions starting from $\widetilde{x}(0)$ denoted by the set $\mathfrak{X}$. Note that, although not explicitly stated, $t_{\max}$ and $\mathcal{I}$ might be different for each solution in $\mathfrak{X}$. We aim to prove that all $\varepsilon_i(t)$ and $\varepsilon_{v_i}(t)$ are bounded and that converge to zero, for all $i \in \mathcal{N}$ and $\widetilde{x}(t) \in \mathfrak{X}$. In view of $\mathcal{D}_c$, for all $\widetilde{x}(t) \in \mathfrak{X}$ it holds that

$$|e_{i,k}(t)| < \bar{\rho}_{i,k}, \quad |e_{v_i,k}(t)| < \bar{\rho}_{v_i,k}, \tag{24}$$

for all $i \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{I}$, where $\bar{\rho}_{i,k}$ and $\bar{\rho}_{v_i,k}$ are the upper bounds of $\rho_{i,k}(t)$ and $\rho_{v_i,k}(t)$, respectively. Consider now the observer variables (19), which can be written as

$$\dot{\hat{v}}_i = -\sum_{i \in \mathcal{N}_j} (\widetilde{v}_i - \widetilde{v}_j) - b_i \widetilde{v}_i, \quad i \in \mathcal{N},$$

where $\widetilde{v}_i := \hat{v}_i - x_{0,2}$, $i \in \mathcal{N}$. Therefore, by setting $\widetilde{v} := \hat{v} - \bar{x}_{0,2} := [\hat{v}_1^\top, \dots, \hat{v}_N^\top]^\top - \bar{x}_{0,2}$, one obtains

$$\dot{\widetilde{v}} = \dot{\hat{v}} = -H\widetilde{v} \tag{25}$$

Since H is positive definite, (25) describes an exponentially stable system. Hence, there exist positive constants q_1 and q_2 such that

$$\|\widetilde{v}(t)\| = \|\hat{v}(t) - \bar{x}_{0,2}(t)\| \leq q_1 \exp(-q_2 t), \forall t \geq 0. \tag{26}$$

Consider now the Lyapunov function $V_p := \frac{1}{2}\varepsilon^\top K_1 \varepsilon$. By differentiating V_p, we obtain in view of (17) - (20) and (24)

$$\begin{aligned}
\dot{V}_p =& \varepsilon^\top K_1 r \rho^{-1}(H(x_2 - \bar{x}_{0,2}) - \dot{\rho}\rho^{-1}e) \\
=& \varepsilon^\top K_1 r \rho^{-1}(H(x_v - \bar{x}_{0,2}) + He_v - \dot{\rho}\rho^{-1}e) \\
=& -\varepsilon^\top r\rho^{-1} K_1 H K_1 \rho^{-1} r \varepsilon + \varepsilon^\top K_1 r \rho^{-1}(H(\widetilde{v} + e_v) \\
& - \dot{\rho}\rho^{-1}e) \\
\leq& -\frac{\underline{H}_1}{M_p}\|r\varepsilon\|^2 + \|r\varepsilon\|\bar{B}_p,
\end{aligned} \tag{27}$$

for all $t \in \mathcal{I}$, where we denote $x_v := [x_{1,v}^\top, \dots, x_{N,v}^\top]^\top$, $e_v := [e_{v_1}^\top, \dots, e_{v_N}^\top]^\top$, and $\bar{B}_p$ is a positive constant satisfying $\bar{B}_p \geq \max_{i \in \mathcal{N}}\{\kappa_{i,1}\}\|\rho^{-1}(H(\widetilde{v} + e_v) - \dot{\rho}\rho^{-1}e)\|$ for all $t \in \mathcal{I}$. Note that $\bar{B}_p$ is independent of $\mathcal{I}$ in view of (24), (26), and the boundedness of $\rho(t)^{-1}$ and $\dot{\rho}(t)$.

Hence, we conclude that $\dot{V}_p < 0$ when $\|r\varepsilon\| > \frac{\bar{B}_p M_p}{\underline{H}_1}$. Since $r_{i,k}$ is positive, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$, the latter is equivalent to $\|\varepsilon\| > \frac{\bar{B}_p M_p}{\underline{H}_1 \underline{r}} \Rightarrow \dot{V}_p < 0$. Hence, we conclude that all $\widetilde{x}(t) \in \mathfrak{X}$ satisfy

$$\|\varepsilon(t)\| \leq \bar{\varepsilon} := \max\left\{\|\varepsilon(0)\|, \frac{\bar{B}_p M_p}{\underline{H}_1 \underline{r}}\right\}. \tag{28}$$

Since $\bar{\varepsilon}$ is finite, it holds that $T(\bar{\varepsilon}) < 1$. Hence $|T(\varepsilon_{i,k}(t))| \leq T(\bar{\varepsilon}) < 1$, for all $i \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{I}$. Moreover, since $T(\cdot)$ and $T^{-1}(\cdot)$ are smooth, the derivative $\frac{\partial T^{-1}(z)}{\partial z}$ approaches infinity only when $z \to \pm 1$. Therefore, in view of the definition of r in (17), we conclude the existence of a finite $\bar{r} > 0$ such that $\|r(t)\| \leq \bar{r}$, for all $t \in \mathcal{I}$. Next, (13) implies that $\|e(t)\| \leq \bar{e} := M_p T(\bar{\varepsilon})\sqrt{n}$, for all $t \in \mathcal{I}$. Hence, we conclude that $\|x_v(t)\| \leq \bar{x}_v := \frac{\max_{i \in \mathcal{N}}\{\kappa_{i,1}\}}{m_p}\bar{r}\bar{\varepsilon}$ and, in view of (12), $\|x_1(t)\| \leq \bar{x}_1 := \frac{\bar{e}}{\Pi(N)} + \sup_{t \geq 0}\|x_{0,1}(t)\|$, for all $t \in \mathcal{I}$. In addition, by employing (24), we conclude that $\|x_2(t)\| < \bar{x}_2 := M_v\sqrt{n} + \bar{x}_v$, for all $t \in \mathcal{I}$. Finally, by differentiating x_v, employing the smoothness and boundedness of ρ and its derivatives, the smoothness of $T(\cdot)$, the boundedness of $x_{0,1}(t)$ and $q_{0,1}$, as well as the aforementioned bounds, we can conclude the existence of a bound $\bar{dx}_v$ such that $\|\dot{x}_v(t)\| \leq \bar{dx}_v$, for all $t \in \mathcal{I}$.

The boundedness of $x(t)$ and Assumption 3 imply the existence of a positive constant $\bar{\zeta}$ such that $\zeta(t) \leq \bar{\zeta}$, for all $t \in \mathcal{I}$. Hence, in view of Assumption 1, and the fact that $\|x_1(t)\| \leq \bar{x}_1$ and $\|x_2(t)\| < \bar{x}_2$, there exists a positive constant $\bar{F}$ such that $\|f(x, \zeta, t)\| \leq \bar{F}$, for all $t \in \mathcal{I}$. We define now the constant

$$d := \frac{\max_{i \in \mathcal{N}}\{\beta_i\}}{m_v}\left(\bar{F} + \bar{dx}_v + M_{\dot{v}}\sqrt{n}\right) + \frac{M_v \bar{H}}{m_p}\bar{r}\bar{\varepsilon} \tag{29}$$

and consider the function

$$V(\widetilde{\varepsilon}) := V_p + \sum_{i \in \mathcal{N}}\left\{\frac{\beta_i}{2}\|\varepsilon_{v_i}\|^2 + \frac{1}{2\gamma_i}\widetilde{d}_i^2\right\},$$

where $\widetilde{\varepsilon} := [\varepsilon^\top, \varepsilon_v^\top, \widetilde{d}]^\top \in \mathbb{R}^{N(2n+1)}$, $\varepsilon_v := [\varepsilon_{v_1}^\top, \dots, \varepsilon_{v_N}^\top]^\top \in \mathbb{R}^{Nn}$, and $\widetilde{d} := [\widetilde{d}_1, \dots, \widetilde{d}_N]^\top := [\hat{d}_1 - d, \dots, \hat{d}_N - d]^\top \in \mathbb{R}^N$; $V(\widetilde{\varepsilon})$ satisfies $W_1(\widetilde{\varepsilon}) \leq V(\widetilde{\varepsilon}) \leq W_2(\widetilde{\varepsilon})$, for $W_1(\widetilde{\varepsilon}) := \min\left\{\frac{1}{2}, \frac{\min_{i \in \mathcal{N}}\{\beta_i\}}{2}, \frac{1}{2\max_{i \in \mathcal{N}}\{\gamma_i\}}\right\}\|\widetilde{\varepsilon}\|^2$ and $W_2(\widetilde{\varepsilon}) := \max\left\{\frac{1}{2}, \frac{\max_{i \in \mathcal{N}}\{\beta_i\}}{2}, \frac{1}{2\min_{i \in \mathcal{N}}\{\gamma_i\}}\right\}\|\widetilde{\varepsilon}\|^2$. Then, according to Lemma 1, $\dot{V}(\widetilde{\varepsilon}(t)) \overset{a.e.}{\in} \dot{\widetilde{V}}(\widetilde{\varepsilon}(t))$ with $\dot{\widetilde{V}} := \cap_{\xi \in \partial V(\widetilde{\varepsilon})}\xi^\top \mathsf{K}\left[\dot{\widetilde{\varepsilon}}\right]$. Since $V(\widetilde{\varepsilon})$ is continuously differentiable, its generalized gradient reduces to the standard gradient and thus it holds that $\dot{\widetilde{V}} = \nabla V^\top \mathsf{K}\left[\dot{\widetilde{\varepsilon}}\right]$, where $\nabla V = [\varepsilon_p^\top, \sum_{i \in \mathcal{N}}\beta_i\varepsilon_{v_i}^\top, \sum_{i \in \mathcal{N}}\frac{1}{\gamma_i}\widetilde{d}_i]^\top$. In view of (27) and (28), the first term of V becomes

$$\begin{aligned}
\dot{V}_p \leq& -\frac{\underline{H}}{M_p}\|r\varepsilon\|^2 + \varepsilon^\top K_1 r \rho^{-1}(H\widetilde{v} - \dot{\rho}\rho^{-1}) + \varepsilon^\top r \rho^{-1} H e_v \\
\leq& -\frac{\underline{H}}{M_p}\|r\varepsilon\|^2 + \bar{r}\bar{\varepsilon}P(t) + \varepsilon^\top r\rho^{-1}He_v
\end{aligned}$$

where $P(t) := \frac{\max_{i \in \mathcal{N}}\{\kappa_{i,1}\}}{m_p}(\bar{H}\|\widetilde{v}(t)\| + \frac{\max_{i \in \mathcal{N}, k \in \mathcal{K}}\{|\dot{\rho}_{i,k}(t)|\}}{m_p})$. By setting $z = T(\varepsilon_{v_i,k})$ in (15), we obtain $|T(\varepsilon_{v_i,k})| \leq |r_{v_i,k}\varepsilon_{v_i,k}|$, $i \in \mathcal{N}$, $k \in \mathcal{K}$ and hence by employing $e_{v_i,k} = \rho_{v_i,k}T(\varepsilon_{v_i,k})$, $i \in \mathcal{N}$, $k \in \mathcal{K}$, we obtain

$$\begin{aligned}
\dot{V}_p \leq& -\frac{\underline{H}}{M_p}\|r\varepsilon\|^2 + \bar{r}\bar{\varepsilon}P(t) + \frac{M_v\bar{H}}{m_p}\bar{r}\bar{\varepsilon}\|r_v\varepsilon_v\| \\
\leq& -\frac{\underline{H}}{M_p}\|r\varepsilon\|^2 + \bar{r}\bar{\varepsilon}P(t) + \frac{M_v\bar{H}}{m_p}\bar{r}\bar{\varepsilon}\sum_{i \in \mathcal{N}}\|r_{v_i}\varepsilon_{v_i}\|
\end{aligned}$$

Therefore, we obtain $\dot{\widetilde{V}} \subset \widetilde{W}_s$, with

$$\begin{aligned}
\widetilde{W}_s :=& \dot{V}_p - \sum_{i \in \mathcal{N}}\beta_i\varepsilon_{v_i}^\top r_{v_i}\rho_{v_i}^{-1}g_i\rho_{v_i}^{-1}\left(\kappa_{i,2}r_{v_i}\varepsilon_{v_i} + \kappa_{i,3}\hat{d}_i\mathsf{S}_{v_i}\right) \\
&+ \sum_{i \in \mathcal{N}}\beta_i\varepsilon_{v_i}^\top r_{v_i}\rho_{v_i}^{-1}\left(f_i - \dot{x}_{i,v} - \dot{\rho}_{v_i}\rho_{v_i}^{-1}e_{v_i}\right) + \sum_{i \in \mathcal{N}}\widetilde{d}_i\|r_{v_i}\varepsilon_{v_i}\|
\end{aligned} \tag{30}$$

Note that, since $\hat{d}_i(0) \geq 0$, (22) implies that $\hat{d}_i(t) \geq 0$, for all $t \in \mathcal{I}$, $i \in \mathcal{N}$. Moreover, since the Filippov regularization is defined as a closed set and $\dot{\widetilde{V}} \subset \widetilde{W}_s$, it holds that $\max_{z\in\dot{\widetilde{V}}}\{z\} \leq \max_{z\in\widetilde{W}_s}\{z\}$. Therefore, after using Assumption 2, the fact that $\beta_i = \frac{1}{\lambda_i \kappa_{i,3}}$, $i \in \mathcal{N}$, and (29), (30) becomes

$$\begin{aligned}
&\max_{z\in\dot{\widetilde{V}}}\{z\} \leq -\frac{H}{M_p}\|r\varepsilon\|^2 - \sum_{i\in\mathcal{N}}\beta_i\kappa_{i,2}\underline{\lambda}_i\|r_{v_i}\varepsilon_{v_i}\|^2 \\
&+ \bar{r}\bar{\varepsilon}P(t) + \sum_{i\in\mathcal{N}}\left\{d\|r_{v_i}\varepsilon_{v_i}\| - \hat{d}_i\|r_{v_i}\varepsilon_{v_i}\| + \widetilde{d}_i\|r_{v_i}\varepsilon_{v_i}\|\right\} \\
&= -\frac{H}{M_p}\|r\varepsilon\|^2 - \sum_{i\in\mathcal{N}}\beta_i\kappa_{i,2}\underline{\lambda}_i\|r_{v_i}\varepsilon_{v_i}\|^2 + \bar{r}\bar{\varepsilon}P(t) \\
&=: -W(\widetilde{\varepsilon}) + \bar{r}\bar{\varepsilon}P(t) \qquad (31)
\end{aligned}$$

where W is a continuous and positive semi-definite function on $\mathbb{R}^{N(2n+1)}$. Moreover, since $\lim_{t\to\infty}\dot{\rho}_{i,k}(t) = -\lim_{t\to\infty} l_{i,k}(\bar{\rho}_{i,k} - \underline{\rho}_{i,k})\exp(-l_{i,k}t) = 0$, $i \in \mathcal{N}$, $k \in \mathcal{K}$, and in view of (26), the term $P(t) = \frac{\max_{i\in\mathcal{N}}\{\kappa_{i,1}\}}{m_p}(\bar{H}\|\widetilde{v}(t)\| + \frac{\max_{i\in\mathcal{N},k\in\mathcal{K}}\{|\dot{\rho}_{i,k}(t)|\}}{m_p})$ converges to zero exponentially. Therefore, there exist positive constants q_3 and q_4 such that (31) becomes

$$\max_{z\in\dot{\widetilde{V}}}\{z\} \leq -W(\widetilde{\varepsilon}) + q_3\exp(-q_4 t), \qquad (32)$$

for all $t \in \mathcal{I}$. Therefore, it holds that $z \leq -W(\widetilde{\varepsilon}) + q_3\exp(-q_4 t)$, for all $z \in \dot{\widetilde{V}}$, $t \in \mathcal{I}$. In addition, note that (32) holds for all the solutions in $\mathfrak{X}$. Choose now any finite $r > 0$ and let $c < \min_{\|\widetilde{\varepsilon}\|=r} W_1(\widetilde{\varepsilon})$. Note that all the conditions of Theorem 2 are satisfied and hence, all Filippov solutions starting from $\widetilde{\varepsilon}(0) \in \Omega_f := \{\widetilde{\varepsilon} \in \mathcal{B}(0,r) : W_2(\widetilde{\varepsilon}) \leq c\}$ are bounded and remain in Ω_f, $\forall t \in \mathcal{I}$. Moreover, $t_{\max} = \infty$, implying that $\mathcal{I} = \mathbb{R}_{\geq 0}$ and it also holds that $\lim_{t\to\infty}\|\varepsilon(t)\| = 0$ and $\lim_{t\to\infty}\|\varepsilon_v(t)\| = 0$, which, in view of the increasing property of $T(\cdot)$ and the fact that $T(0) = 0$, implies that $\lim_{t\to\infty}\|e(t)\| = 0$ and $\lim_{t\to\infty}\|e_v(t)\| = 0$.

Note that r, and hence c, can be arbitrarily large allowing any finite initial condition $\widetilde{\varepsilon}$, which implies any $(\widetilde{x}(0), 0) \in \mathcal{D}_c$. In addition, it holds that $\|\widetilde{\varepsilon}\|^2 \leq \widetilde{c} := \min\left\{\frac{1}{2}, \frac{\min_{i\in\mathcal{N}}\{\beta_i\}}{2}, \frac{1}{2\max_{i\in\mathcal{N}}\{\gamma_i\}}\right\}^{-1} c$, which implies the boundedness of $\|\varepsilon\|$, $\|\varepsilon_v\|$ and $\widetilde{d}$ by $\sqrt{\widetilde{c}}$. Therefore, we conclude that $\|\hat{d}(t)\| \leq \bar{d} := d + \sqrt{\widetilde{c}}$ and that all solutions are bounded in compact sets for all $t \in \mathcal{I}$, which means that u and $\dot{\hat{d}}$, as designed in (21) and (22), respectively, remain also bounded, for all $t \in \mathcal{I}$. ■

Remark 3: Note that no boundedness assumptions or growth conditions are needed for the unknown vector fields f_i and g_i, $i \in \mathcal{N}$. Moreover, the response of the system is solely determined by the funnel functions $\rho_{i,k}$ and $\rho_{v_i,k}$, isolated from the system dynamics and the control gains. Nevertheless, appropriate gain tuning might be needed in order to suppress chattering due to the discontinuous nature of the proposed algorithm. Finally, unlike most related works, $\rho_{i,k}$ are not required to decrease to values arbitrarily close to zero, and *asymptotic stability* is still achieved.

V. SIMULATION RESULTS

We perform simulations with a leader and 4 follower agents with $b_1 = 1$, $b_2 = b_3 = b_4 = 0$, $a_{ij} = 1$ for $(i,j) \in \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,4),(4,3)\}$ and 0 otherwise. The leader evolves according to $x_{0,1} = [2t, -t]^\top$ and each follower consists of two inverted pendulum connected by a spring and a damper, with dynamics:

$$\begin{aligned}
J_{i,1}\ddot{x}_{i,1} &= g_s\sin(x_{i,1}) - 0.25F_{s,i}\cos(x_{i,1} - \theta_i) - T_{i,1} + u_{i,1} \\
J_{i,2}\ddot{x}_{i,2} &= 1.25g_s\sin(x_{1,2}) + 0.25F_{s,i}\cos(x_{i,2} - \theta_i) - T_{i,2} + \sigma(t)u_{i,2},
\end{aligned}$$

where $F_{s,i} := 150(d_{s,i} - 0.5) + \dot{d}_{s,i}$ is the force between the connection points of the spring and damper, and

$$d_{s,i} := \sqrt{.25 + .25(\sin(x_{i,1} - x_{i,2})) + 0.125(1 - \cos(x_{i,2} - x_{i,1}))}$$

is the distance between these points; θ_i is defined as

$$\theta_i := \tan^{-1}\left(\frac{0.25(\cos(x_{i,2}) - \cos(x_{i,1}))}{0.5 + 0.25(\sin(x_{i,1}) - \sin(x_{i,2}))}\right)$$

and $T_{i,1}$, $T_{i,2}$ are friction terms on the motors evolving according to $T_{i,j} = d_{a_j,i}(t) + \tau_{i,j} + \dot{\tau}_{i,j} + \dot{x}_{i,j}$, with

$$\dot{\tau}_{i,j} = \dot{x}_{i,j} - |\dot{x}_{i,j}|\left(1 + \exp\left(-\left|\frac{\dot{x}_{i,j}}{0.1}\right|^2\right)\right)^{-1}$$

and $d_{a_j,i}(t) := (-1)^{j-1}\cos(t)^2$, for $t \in \left[0, \frac{3\pi}{2}\right) \cup \left[\frac{7\pi}{2}, \frac{11\pi}{2}\right) \cup \left[\frac{21\pi}{2}, \frac{27\pi}{2}\right) \cup \left[\frac{35\pi}{2}, 50\right)$ and 0 otherwise, $j \in \{1,2\}$, being an additional disturbance. The time varying signal $\sigma(t)$ is set as $\sigma(t) = 1$ if $t \in [0,3) \cup [3.5,\infty)$, and $\sigma(t) = 0.5$ if $t \in [3,3.5)$, modeling a loss of effectiveness of the second motors when $t \in [3,3.5)$. We also choose $g_s = 9.81$ as the gravity constant and $J_{i,1} = 0.5$, $J_{i,2} = 0.625$. The initial conditions are $x_1(0) = [-0.7606, -0.2699]^\top$, $x_2(0) = [-1.2288, -1.6145]^\top$, $x_3(0) = [0.8140, -0.8459]^\top$, $x_4(0) = [1.9924, -1.9092]^\top$, $\tau_{i,1}(0) = \tau_{i,2}(0) = 0$ for all $i \in \{1,\ldots,4\}$. The prescribed funnel functions are chosen as $\rho_{i,1}(t) = \rho_{i,2}(t) = 2.5\exp(-0.1t) + 2.5$, $\forall i \in \{1,\ldots,4\}$, which converge to 2.5. We also choose $\rho_{v_i,1}(t) = \rho_{v_i,2}(t) = (\|e_{v_i}(0)\| - 2)\exp(-0.1t) + 2.5$, as well as the gains $\kappa_{i,1} = 100$, $\kappa_{i,2} = 2 \cdot 10^3$, $k_{i,3} = 0.0125$, and $\gamma_i = 50$ for all $i \in \mathcal{N}$. The simulation results are depicted in Figs. 1-2 for $t \in [0,25]$ sec. More specifically, Fig. 1 depicts the synchronization errors $e_{i,k}(t)$ along with the performance functions $\rho_{i,k}(t)$, for $i \in \{1,\ldots,4\}$ and $k \in \{1,2\}$. One can conclude that $e_{i,k}(t)$ not only respect their imposed funnels but also converge asymptotically to zero, without the need of arbitrarily small values for $\lim_{t\to\infty}\rho_{i,k}(t)$. Finally, Figs. 2 depicts the adaptation variables $\hat{d}_i(t)$ and control inputs $u_i(t)$ for $i \in \mathcal{N}$. One can conclude the convergence of $\hat{d}_i(t)$ to constant values as well as the boundedness of the control input $u_i(t)$, as proved in the theoretical analysis.

VI. CONCLUSION AND FUTURE WORK

This paper presents a distributed control algorithm that guarantees asymptotic synchronization subject to funnel constraints for a class of 2nd-order multi-agent systems with unknown, nonlinear dynamics. Future efforts will be devoted towards extending the proposed scheme to more general systems, directed graphs, and controllability relaxations.

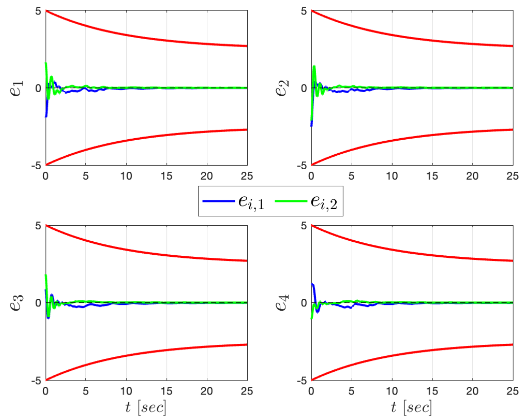

Fig. 1. The evolution of the errors $e_{i,k}(t)$, depicted with blue and green, along with the performance functions $\rho_{i,k}(t)$, depicted with red, for $i \in \{1, \ldots, 4\}$, $k \in \{1, 2\}$, and $t \in [0, 25]$ sec.

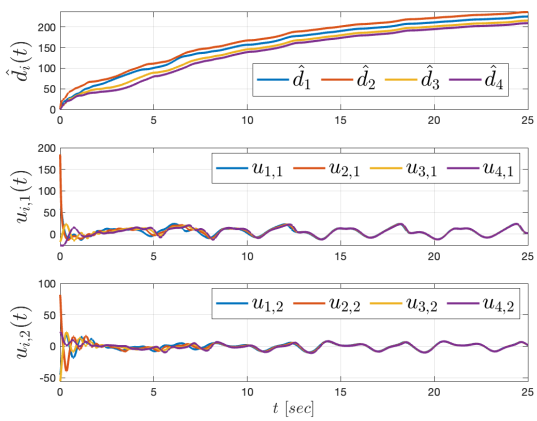

Fig. 2. The evolution of the adaptation variables $\hat{d}_i(t)$ (top) and the control inputs $u_i(t)$ (middle and bottom) for $i \in \{1, \ldots, 4\}$ and $t \in [0, 25]$ sec.

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