# Verifiable and Compositional Reinforcement Learning Systems

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## Abstract

We propose a novel framework for verifiable and compositional reinforcement learning (RL) in which a collection of RL sub-systems, each of which learns to accomplish a separate sub-task, are composed to achieve an overall task. The framework consists of a *high-level* model, represented as a parametric Markov decision process (pMDP) which is used to plan and to analyze compositions of sub-systems, and of the collection of *low-level* sub-systems themselves. By defining interfaces between the sub-systems, the framework enables automatic decompositons of task specifications, *e.g., reach a target set of states with a probability of at least 0.95*, into individual sub-task specifications, *i.e. achieve the sub-system's exit conditions with at least some minimum probability, given that its entry conditions are met*. This in turn allows for the independent training and testing of the sub-systems; if they each learn a policy satisfying the appropriate sub-task specification, then their composition is guaranteed to satisfy the overall task specification. Conversely, if the sub-task specifications cannot all be satisfied by the learned policies, we present a method, formulated as the problem of finding an optimal set of parameters in the pMDP, to automatically update the sub-task specifications to account for the observed shortcomings. The result is an iterative procedure for defining sub-task specifications, and for training the sub-systems to meet them. As an additional benefit, this procedure allows for particularly challenging or important components of an overall task to be determined automatically, and focused on, during training. Experimental results demonstrate the presented framework's novel capabilities. A collection of RL sub-systems are trained, using proximal policy optimization algorithms, to navigate different portions of a labyrinth environment. A cross-labyrinth task specification is then decomposed into sub-task specifications. Challenging portions of the labyrinth are automatically avoided if their corresponding sub-systems cannot learn satisfactory policies within allowed training budgets. Other unnecessary sub-systems are not trained at all. The result is a compositional RL system that efficiently learns to satisfy its task specification.

## 1 Introduction

Reinforcement learning (RL) algorithms offer tremendous capabilities in systems that work with unknown environments. However, there remain significant barriers to their deployment in safety-critical engineering applications. Autonomous vehicles, manufacturing robotics, and power systems management are examples of complex application domains that require strict adherence of the system's behavior to stakeholder requirements. However, the verifiation of RL systems is difficult. This is particularly true of monolithic end-to-end RL approaches; many model-free RL algorithms, for instance, only output the learned policy and its estimated value function, rendering them opaque for

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Preprint. Under review.

verification purposes. The difficulty of verification is compounded in engineering application domains, which often require large observation and action spaces, and complicated reward functions.

How do we build complex engineering systems we can trust? Engineering design principles have long prescribed system modularity as a means to reduce the complexity of individual sub-systems [9, 20]. By creating well-defined interfaces between sub-systems, system-level requirements may be decomposed into component-level ones. Conversely, each component may be developed and tested independently, and the satisfaction of component-level requirements may then be used to place assurances on the behavior of the system as a whole. Building RL systems that incorporate such engineering practices and guarantees is a crucial step toward their widespread deployment [16].

Toward this end, we develop a framework for verifiable and compositional reinforcement learning. The framework comprises two levels of abstraction. The *high level* is used to plan *meta-policies* and to verify their adherence to task specifications, *e.g., reach a particular goal state with a probability of at least 0.9*. Meta-policies dictate sequences of *sub-systems* to execute, each of which is designed to accomplish a specific *sub-task*, *i.e. achieve a particular exit condition, given the sub-system is executed from one of its entry conditions*. We assume a collection of *partially instantiated* sub-systems to be given a priori; their entry and exit conditions are known, but the policies they implement are not. These entry and exit conditions might be defined by pre-existing engineering capabilities, explicitly by a task designer, or by entities within the environment. At the *low level* of the framework, each sub-system employs RL algorithms to learn policies accomplishing its sub-task.

We model the high level of the framework using a parametric Markov decision process (pMDP) [5, 13]. Each action in the pMDP represents an individual RL sub-system, and the parametric transition probabilities in the pMDP thus represent the likelihoods of outcomes that could occur when the sub-system is executed. Using sampling-based estimates of sub-system policies, we assign values to the model parameters and use existing MDP techniques for the planning and verification of meta-policies [21, 1]. Beyond this capability, the presented framework offers the following novel features.

**1. Automatic decomposition of task specifications.** We formulate, as the problem of finding an optimal set of parameters in the pMDP, a method to automatically decompose the task specification into sub-task specifications, allowing for independent learning and verification of the sub-systems.

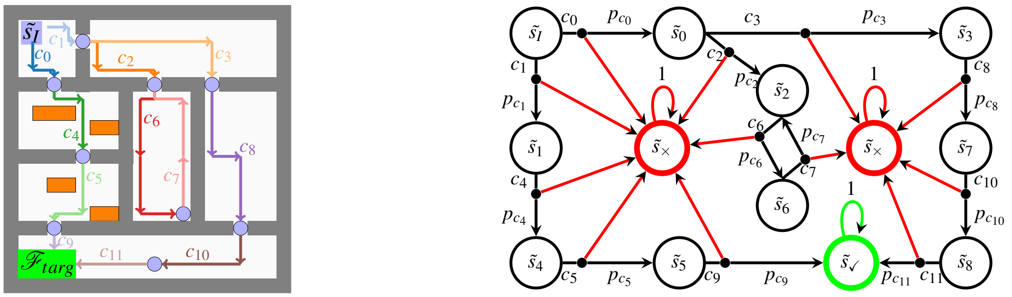
**2. Learning to satisfy sub-task specifications.** Any RL method can be used to learn the sub-system policies, so long as the learned policies satisfy the relevant sub-task specification. We present a sub-system reward function definition, in terms of the exit conditions of the sub-system, that motivates the learning of policies satisfying the sub-task specification. Furthermore, these sub-task specifications provide an *interface* between the sub-systems, allowing for the analysis of their compositions. In particular, we guarantee that if each of the learned sub-system policies satisfies its sub-task specifications, a composition of them exists satisfying the specifications on the overall task.

**3. Iterative specification refinement.** However, if some of the sub-task specifications cannot be satisfied by the corresponding learned policies, sampling-based estimates of their behavior are used to update the high-level model. We present a method to use this information to refine the sub-task specifications, in order to better reflect what might realistically be achieved by the sub-systems. This automatic refinement naturally leads to a compositional RL algorithm that iteratively computes sub-task specifications, and then trains the corresponding sub-systems to achieve them.

**4. System modularity: prediction and verification in task transfer.** By providing an interface between the sub-tasks, the presented framework allows for previously learned sub-task policies to be re-used as components of new high-level models, designed to solve different tasks. Furthermore, the sub-task specifications themselves may be re-used to perform verification within these new models, without the need for further training.

Experimental results exemplify these novel capabilities in a labyrinth navigation task environment. We use proximal policy optimization algorithms [24] to train individual sub-systems to navigate portions of the environment, which are then composed to complete a cross-labyrinth navigation task. Through the aforementioned compositional RL algorithm, the task specification is decomposed and challenging portions of the labyrinth are avoided if their corresponding sub-systems cannot learn to satisfy the relevant sub-task specification.

**Related Work.** The proposed multi-layered task abstraction resembles hierarchical RL (HRL) [25, 2, 14, 27, 17]. HRL methods reduce computational complexity, particularly in problems with large



(a) The labyrinth task environment.

(b) The HLM corresponding to the labyrinth example.

Figure 1: An example labyrinth navigation task. Figure (a) illustrates the environment, as well as an example collection of sub-systems, represented by the colored paths. Entry and exit conditions for the various sub-systems are shown as blue circles. Figure (b) illustrates the corresponding HLM. Each sub-system $c$ causes a transition to its successor state with probability $p_c$. Otherwise, the HLM transitions to the failure state $\tilde{s}_\times$ with probability $1 - p_c$, visualized by the red transitions.

state and action spaces. However, they typically focus on the efficient maximization of discounted reward, and require the meta-policy be learned; no model of the high-level problem is explicitly constructed. As such, these methods are unable to answer questions pertaining to the verification of task specifications. By contrast, we present a framework with the specific aim of enabling verifiable RL against a rich set of specifications, while enjoying a similar reduction in sample complexity.

Compositional verification has been studied in formal methods [18, 7], but not in the context of RL. Conversely, some recent works have used structured task knowledge to decompose RL problems, however, they do not study how such information can be used for the verification and automatic decomposition of task specifications. [3, 15] define a task specification language based on linear temporal logic, and subsequently use it to generate reward functions for RL. [23] incorporates RL with symbolic planning models to learn new operators – similar to our sub-tasks – to aid in the completion of planning objectives. Meanwhile, [12, 26, 28, 11] use reward machines, finite-state machines encoding temporally extended tasks, to break tasks into stages for which separate policies can be learned. [19] extends the use of reward machines to the multi-agent RL setting, decomposing team tasks into sub-tasks for individual learners.

**Outline.** In §2, we define the notions of tasks, specifications, and systems, which are necessary for the development of the proposed framework. In §3, we introduce the *high-level model* (HLM), and we present how it can be used to plan policies and to automatically decompose the task specification into separate specifications for the individual sub-tasks. In §4 we discuss how to learn policies satisfying the sub-tasks specifications, as well as how empirical rollouts of existing policies can be used to update the HLM, and refine the sub-task specifications. We then present a novel compositional RL algorithm that alternates between computing sub-task specifications, and learning sub-system policies to satisfy them. Finally, experimental results are presented in §5.

## 2 The Compositional Reinforcement Learning Framework

To begin introducing the proposed framework, and to provide intuitive examples of the notions of tasks, sub-tasks, systems, and sub-systems, we consider the example shown in Figure 1a. The figure illustrates a labyrinth environment, which is composed of a collection of interconnected rooms. The *system* executes its constituent *sub-systems* in this environment to complete an overall task. The *task* is to safely navigate from the labyrinth's initial state in the top left corner to the goal state marked by a green square in the bottom left corner. Satisfaction of the *task specification* requires that the system successfully completes the task with a probability of at least 0.95. As an added difficulty, lava exists within some of the rooms, represented in the figure by the orange rectangles. If the lava is touched, the task is automatically failed. This task is naturally decomposed into separate *sub-tasks*, each of which navigates an individual room, and is executed by a separate sub-system.

**Preliminaries.** We model the task environment as a Markov decision process (MDP), which is defined by a tuple $M = (S, A, P)$. Here, $S$ is a set of states, $A$ is a set of actions, and $P : S \times A \times S \rightarrow [0,1]$ is a transition probability function. A stationary policy $\pi$ within the MDP is a function $\pi : S \times A \rightarrow [0,1]$ such that $\sum_{a \in A} \pi(s,a) = 1$ for every $s \in S$. Intuitively, $\pi(s,a)$ assigns the probability of taking action $a$ from state $s$ under policy $\pi$. Given an MDP $M$, a policy $\pi$, and a target set of states $S_{targ} \subseteq S$, we define $\mathbb{P}^{s}_{M,\pi}(\Diamond S_{targ})$ to be the probability of eventually reaching some state $s' \in S_{targ}$, beginning from the initial state $s$, under policy $\pi$. Similarly, $\mathbb{P}^{s}_{M,\pi}(\Diamond_{\leq T} S_{targ})$ denotes the probability of reaching the target set from state $s$ within some finite time horizon $T$.

**RL Sub-Systems and Sub-Tasks.** We define each RL sub-system $c$ acting within the environment by the tuple $c = (\mathscr{I}_c, \mathscr{F}_c, T_c, \pi_c)$. Here, $\mathscr{I}_c \subseteq S$ is a set defining the sub-system's *entry conditions*, $\mathscr{F}_c \subseteq S$ is a set representing the sub-system's *exit conditions*, and $T_c \in \mathbb{N}$ is the sub-system's allowed *time horizon*. The *sub-task* associated with each sub-system, is to navigate from any entry condition $s \in \mathscr{I}_c$ to any exit condition $s' \in \mathscr{F}_c$ within the sub-system's time horizon $T_c$. We assume that each sub-system may only be *executed*, or begun, from an entry condition $s \in \mathscr{I}_c$ and that its execution ends either when it achieves an exit condition $s \in \mathscr{F}_c$, or when it runs out of time. Finally, $\pi_c : S \times A \to [0,1]$ is the policy that the component implements to complete this objective. For notational convenience, we define $\sigma^{c}_{\pi_c}(s) := \mathbb{P}^{s}_{M,\pi_c}(\Diamond_{\leq T_c} \mathscr{F}_c)$. A *sub-task specification*, is then defined as the requirement that $\sigma^{c}_{\pi_c}(s) \geq p_c$ for every entry condition $s \in \mathscr{I}_c$ of the sub-system. Here, $p_c \in [0,1]$ is a value representing the minimum allowable probability of the sub-task success. We note that such reachability-based task specifications are very expressive. Temporal logic specifications can be expressed as reachability specifications in a so-called product MDP [1, 10].

We say a sub-system $c$ is *partially instantiated* when $\mathscr{I}_c$, $\mathscr{F}_c$, and $T_c$ are defined, but its policy $\pi_c$ is not. We define a collection $\mathscr{C} = \{c_1, c_2, ..., c_k\}$ of sub-systems to be *composable*, if and only if for every $i, j \in \{1, 2, \ldots, k\}$, either $\mathscr{F}_{c_i} \subseteq \mathscr{I}_{c_j}$ or $\mathscr{F}_{c_i} \cap \mathscr{I}_{c_j} = \emptyset$. In words, sub-systems are composable when the set of exit conditions of each sub-system is a subset of all the sets of entry conditions that it intersects. This ensures that regardless of the specific exit condition $s \in \mathscr{F}_c$ in which sub-system $c$ terminates, $s$ will be a valid entry condition for the *same* collection of other sub-systems.

**Compositions of RL Sub-Systems.** Compositions of sub-systems are specified by *meta-policies* $\mu : S \times \mathscr{C} \to [0,1]$, which assign probability values to the execution of different sub-systems, given the current environment state $s \in S$. So, execution of the composite system occurs as follows. From a given state $s$, the meta-policy is used to select a sub-system $c$ to execute. The sub-system's policy $\pi_c$ is then followed until it either reaches an exit condition $s' \in \mathscr{F}_c$, or it reaches the end of its time horizon $T_c$. If the former is true, the meta-policy selects the next sub-system to execute from $s'$, and the process repeats. Conversely, if the latter is true, the sub-system has failed to complete its sub-task in time, and the execution of the meta-policy stops. In the labyrinth example, the meta-policy selects which rooms to pass through, while the sub-systems policies navigate the individual rooms.

The *task* of the composite system is, beginning from an initial state $s_I$, to eventually reach a particular target exit condition $\mathscr{F}_{targ} \subseteq S$. We assume that $\mathscr{F}_{targ}$ is equivalent to $\mathscr{F}_c$ for at least one of the sub-systems. That is, there is some sub-system $c \in \mathscr{C}$ such that $\mathscr{F}_{targ} = \mathscr{F}_c$. Furthermore, to simplify theoretical analysis, we assume that for every $c \in \mathscr{C}$, either $\mathscr{F}_c = \mathscr{F}_{targ}$ or $\mathscr{F}_c \cap \mathscr{F}_{targ} = \emptyset$. This assumption removes ambiguity as to whether or not completion of a given sub-task results in the immediate completion of the system's task. Finally, we assume that at least one sub-system $c$ can be executed from the initial state $s_I$, i.e. there exists a sub-system $c \in \mathscr{C}$ such that $s_I \in \mathscr{I}_c$. We say that the execution of a meta-policy reaches the target set $\mathscr{F}_{targ}$, when one of the sub-systems $c$ with $\mathscr{F}_c = \mathscr{F}_{targ}$ is executed, and successfully completes its sub-task. With a slight abuse of notation, we denote the probability of eventually reaching the target set under meta-policy $\mu$ by $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ})$.

A *task specification* places a requirement on the probability of the compositional RL system reaching $\mathscr{F}_{targ}$. That is, for some allowable failure probability $\delta \in [0,1]$, the task specification is satisfied if $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ}) \geq 1 - \delta$. With these definitions in place, we now deliver our problem statement.

**Problem Statement.** *Given an allowable failure probability $\delta \in [0,1]$, an initial state $s_I$, a target set $\mathscr{F}_{targ}$, and a partially instantiated collection $\mathscr{C}$ of composable sub-systems, learn policies $\pi_c$ for each sub-system $c \in \mathscr{C}$ and compute a corresponding meta-policy $\mu$ such that $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ}) \geq 1 - \delta$.*

## 3 The High-Level Decision-Making Model

We now introduce the high-level model (HLM) of the compositional RL framework, which is used to compute meta-policies, and to decompose task specifications into separate sub-task specifications to be satisfied by the individual sub-systems.

**Defining the High-Level Model (HLM).** To construct the HLM, we use a given collection $\mathscr{C} = \{c_1, c_2, \dots, c_k\}$ of partially instantiated sub-systems, an initial state $s_I$, and a target set $\mathscr{F}_{targ}$. We begin by defining a state abstraction, which groups together environment states in order to define the state space of the HLM. To do so, we define the equivalence relation $R \subseteq S \times S$ as follows.

$$(s, s') \in R \text{ if and only if } \begin{cases} \text{1. For every } c \in \mathscr{C}, s \in \mathscr{I}_c \text{ if and only if } s' \in \mathscr{I}_c, \text{ and,} \\ \text{2. } s \in \mathscr{F}_{targ} \text{ if and only if } s' \in \mathscr{F}_{targ}. \end{cases}$$

The equivalence class of any state $s \in S$ under equivalence relation $R$ is given by $[s]_R = \{s' \in S | (s, s') \in R\}$. The quotient set of $S$ by $R$ is defined as the set of all equivalence classes $S/_R = \{[s]_R | s \in S\}$. Intuitively, this equivalence relation groups together all the states in the target set, and it also groups together states that are entry conditions to the same subset of sub-systems.

We may now define the HLM corresponding to the collection $\mathscr{C}$ by the parametric MDP $\tilde{M} = (\tilde{S}, \tilde{s}_I, \tilde{s}_{\checkmark}, \tilde{s}_{\times}, \mathscr{C}, \tilde{P})$. Here, the high-level states $\tilde{S}$ are defined to be $S/_R$; states in the HLM correspond to equivalence classes of environment states. The initial state $\tilde{s}_I$ of the HLM is defined as $\tilde{s}_I = [s_I]_R$, the equivalence class of the environment's initial state. The *goal state* $\tilde{s}_{\checkmark} \in \tilde{S}$ is similarly defined as $[s]_R$ such that $s \in \mathscr{F}_{targ}$. Recall that $\mathscr{F}_{targ} = \mathscr{F}_c$ for at least one of the sub-systems $c \in \mathscr{C}$. Finally, the*failure state* $\tilde{s}_{\times} \in \tilde{S}$ is defined as $[s]_R$ such that $s \in S \setminus [\bigcup_{c \in \mathscr{C}} \mathscr{I}_c] \cup \mathscr{F}_{targ}$, i.e., the equivalence class of states *not* belonging to the initial states of any component, or to the target set.

As an example, Figure 1b illustrates the HLM corresponding to the collection of sub-systems from Figure 1a. The overlapping entry and exit conditions, represented by the blue circles in Figure 1a, define the states of the HLM. The target set $\mathscr{F}_{targ}$ defines the HLM's goal state $\tilde{s}_{\checkmark}$, and all the other environment states are absorbed into the HLM's failure state $\tilde{s}_{\times}$.

The collection of sub-systems $\mathscr{C}$ defines the HLM's set of actions. Note that by definition of the equivalence relation $R$, for every HLM state $\tilde{s} \in \tilde{S}$ there is a well-defined subset of the sub-systems $\mathscr{C}(\tilde{s}) \subseteq \mathscr{C}$ that can be executed. That is, for every environment state $s \in \tilde{s}$, $s \in \mathscr{I}_c$ for all $c \in \mathscr{C}(\tilde{s})$. We define $\mathscr{C}(\tilde{s})$ to be the set of *available sub-systems* at high-level state $\tilde{s}$.

Furthermore, consider any sub-system $c \in \mathscr{C}(\tilde{s})$. As a direct result of the definition of equivalence relation $R$ and of the sub-systems in collection $\mathscr{C}$ being composable, every state $s$ within set $\mathscr{F}_c$ belongs to the *same* equivalence class $[s]_R$. In other words, we may uniquely define the successor HLM state of any component $c \in \mathscr{C}$ as $succ(c) = [s]_R$ such that $s \in \mathscr{F}_c$. We then construct the HLM transition probability function in terms of parameters $p_c \in [0, 1]$ as follows.

$$\tilde{P}(\tilde{s}, c, \tilde{s}') = \begin{cases} p_c, & if \;\; c \in \mathscr{C}(\tilde{s}), \;\; \tilde{s}' = succ(c) \\ 1 - p_c, & if \;\; c \in \mathscr{C}(\tilde{s}), \;\; \tilde{s}' = \tilde{s}_{\times} \\ 0, & \text{Otherwise} \end{cases}$$

The interpretation of this definition of $\tilde{P}$ is as follows. After selecting component $c \in \mathscr{C}(\tilde{s})$ from HLM state $\tilde{s}$, the component either succeeds in reaching an exit condition $s \in \mathscr{F}_c$ within its time horizon $T_c$ with probability $p_c$, resulting in an HLM transition to $succ(c)$, or it fails to do so with probability $1 - p_c$, resulting in a transition to the HLM failure state $\tilde{s}_{\times}$.

The parameters $p_c$ may thus be interpreted as estimates of the probabilities that the sub-systems complete their sub-tasks, given they are executed from one of their entry conditions. Their values come either from empirical rollouts of learned sub-system policies $\pi_c$, or as the solution to the aforementioned automatic decomposition of the task specification, which is discussed further below.

**Relating the HLM to Compositions of RL Sub-Systems.** We note that while parameters $p_c$ are meant to estimate the probabilities of successful sub-task completion, they cannot capture these probabilities exactly. In reality, while parameter $p_c$ is constant, its possible for this probability to vary, given the entry condition $s \in \mathscr{I}_c$ from which the component is executed. However, the simplicity of the presented parametrization of $\tilde{P}$ enables tractable solutions to planning and verification problems in $\tilde{M}$. Furthermore, by establishing relationships between policies in $\tilde{M}$, and meta-policies composing RL sub-systems, the HLM becomes practically useful in the analysis of composite RL systems.

Towards this idea, we note that any stationary policy $\tilde{\mu} : \tilde{S} \times \mathscr{C} \to [0, 1]$ acting in HLM $\tilde{M}$ defines a unique compositional meta-policy $\mu : S \times \mathscr{C} \to [0, 1]$ as follows: for any environment state $s$ and component $c$, define $\mu(s, c) := \tilde{\mu}([s]_R, c)$. So, solutions to planning problems in $\tilde{M}$ can be used directly as meta-policies to specify compositions of the RL sub-systems. Of particular interest, is the problem of computing an HLM policy $\tilde{\mu}$ that maximizes $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond \tilde{s}_\checkmark)$, the probability of eventually reaching the goal state $\tilde{s}_\checkmark$ from the HLM's initial state $\tilde{s}_I$. Theorem 1 relates this probability to the corresponding meta-policy's probability of completing its task, $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ})$, in the environment.

**Theorem 1.** *Let $\mathscr{C} = \{c_1, c_2, ..., c_k\}$ be a collection of composable sub-systems with respect to initial state $s_I$ and target set $\mathscr{F}_{targ}$ within the environment MDP $M$. Define $\tilde{M}$ to be the corresponding HLM and let $\tilde{\mu}$ be a policy in $\tilde{M}$. If, for every sub-system $c \in \mathscr{C}$ and for every entry condition $s \in \mathscr{I}_c$, $\sigma^c_{\pi_c}(s) \geq p_c$, then $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ}) \geq \mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond \tilde{s}_\checkmark)$.*

For example, consider the labyrinth task from Figure 1a, and its corresponding HLM from Figure 1b. Suppose the HLM's parameters $p_c$ are specified such that they lower bound the true probabilities of sub-task success, i.e. the transition probabilities in Figure 1b lower bound the probabilities of the sub-systems successfully navigating their respective rooms in Figure 1a. By planning a policy $\tilde{\mu}$ in the HLM that, for example, reaches $\tilde{s}_\checkmark$ with probability 0.95, we ensure that the corresponding composition of the sub-systems will reach $\mathscr{F}_{targ}$ in the labyrinth with a probability of *at least* 0.95.

**Automatic Decomposition of Task Specifications.** Recall that our objective is not only to compute a meta-policy $\mu$, but also to *learn* the sub-system policies $\pi_{c_1}, \pi_{c_2}, ..., \pi_{c_k}$ that this meta-policy will execute, such that the system's task specification $\mathbb{P}^{s_I}_{M,\mu}(\Diamond \mathscr{F}_{targ}) \geq 1 - \delta$ is satisfied. Suppose that we choose a set of HLM parameters $\{p_{c_1}, p_{c_2}, ..., p_{c_{c_k}}\}$ such that a policy $\tilde{\mu}$ in the HLM exists with $\mathbb{P}^{s_I}_{M,\mu}(\Diamond S_{targ}) \geq 1 - \delta$. Then, so long as each of the corresponding sub-systems $c$ are able to learn a policy $\pi_c$ such that $\sigma^c_{\pi_c}(s) \geq p_c$ for every $s \in \mathscr{I}_c$, Theorem 1 tells us that the meta-policy defined by $\mu(s,c) := \tilde{\mu}([s]_R, c)$ is guaranteed to satisfy the task specification $\mathbb{P}^{s_I}_{M,\mu}(\Diamond S_{targ}) \geq 1 - \delta$.

We may thus interpret the values of parameters $p_c$ as *sub-task specifications*: each sub-system must achieve one of its exit conditions $s' \in \mathscr{F}_c$ within its allowed time horizon $T_c$ with a probability of at least $p_c$, given its execution began from some entry condition $s \in \mathscr{I}_c$. With this interpretation in mind, we take the following approach to the decomposition of the task specification: find the smallest values of parameters $p_{c_1}, p_{c_2}, ..., p_{c_k}$ such that an HLM policy $\tilde{\mu}$ exists satisfying $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond \tilde{s}_\checkmark) \geq 1 - \delta$. We formulate this constrained parameter optimization problem as the bilinear program given in equations (1)-(4). In (2) and (4), we define $pred(\tilde{s}) := \{(\tilde{s}', c') | c' \in \mathscr{C}(\tilde{s}') \text{ and } \tilde{s} \in succ(c')\}$.

$$\min_{x, p_c} \quad \textstyle\sum_{c \in \mathscr{C}} p_c \tag{1}$$

$$\text{s.t.} \quad \textstyle\sum_{c \in \mathscr{C}(\tilde{s})} x(\tilde{s}, c) = \delta_{\tilde{s}_I}(\tilde{s}) + \sum_{(\tilde{s}', c') \in pred(\tilde{s})} x(\tilde{s}', c') p_{c'}, \ \forall \tilde{s} \in \tilde{S} \setminus \{\tilde{s}_\times, \tilde{s}_\checkmark\} \tag{2}$$

$$x(\tilde{s}, c) \geq 0, \ \forall \tilde{s} \in \tilde{S} \setminus \{\tilde{s}_\times, \tilde{s}_\checkmark\}, \ \forall c \in \mathscr{C}(\tilde{s}), \qquad 0 \leq p_c \leq 1, \ \forall c \in \mathscr{C} \tag{3}$$

$$\textstyle\sum_{(\tilde{s}', c') \in pred(\tilde{s}_\checkmark)} x(\tilde{s}', c') p_{c'} \geq 1 - \delta \tag{4}$$

The decision variables in (1)-(4) are the HLM parameters $p_c$ for every $c \in \mathscr{C}$, and $x(\tilde{s}, c)$ for every $\tilde{s} \in \tilde{S} \setminus \{\tilde{s}_\times, \tilde{s}_\checkmark\}$. The value of $\delta_{\tilde{s}_I}(\tilde{s})$ is 1 if $\tilde{s} = \tilde{s}_I$ and 0 otherwise. The constraint (2) is the so-called Bellman-flow constraints and ensures that the variable $x(\tilde{s}, c)$ defines the expected number of times sub-system $c$ is executed in state $\tilde{s}$. The constraint (4) enforces the HLM policy $\tilde{\mu}$'s satisfaction of $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond \tilde{s}_\checkmark) \geq 1 - \delta$. We refer to [6] and [21] for further details on these variables and the constraints.

## 4 Iterative Compositional Reinforcement Learning (ICRL)

In this section, we discuss how sub-system policies are learned to satisfy the sub-task specifications obtained in §3, and we present how the bilinear program given in (1)-(4) is modified to refine the sub-task specifications, after some training of the sub-systems has been completed.

**Learning and Verifying Sub-System Policies.** Let $p_{c_1}, p_{c_2}, ..., p_{c_k}$ be the parameter values output as a solution to problem (1)-(4). We want each sub-system $c$ to learn a policy $\pi_c$ satisfying the sub-task specification: $\sigma^c_{\pi_c}(s) \geq p_c$ for each entry condition $s \in \mathscr{I}_c$ of the sub-system. We note that any RL algorithm and reward function may be used, so long as the resulting learned policy can be verified to satisfy its sub-task specification. A particularly simple candidate reward function $R_c$ outputs 1 when an exit condition $s \in \mathscr{F}_c$ is first reached, and outputs 0 otherwise. Under this reward

**Algorithm 1:** Iterative Compositional Reinforcement Learning

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Input: Partially instantiated sub-systems $\mathscr{C} = \{c_1, c_2, ..., c_k\}$, $\delta$, $N_{train}$, $N_{max}$.
Output: Sub-system policies $\{\pi_{c_1}, \pi_{c_2}, ..., \pi_{c_k}\}$, meta-policy $\mu$.
1  $\tilde{M} \leftarrow ConstructHLM(\mathscr{C})$
2  $\{\pi_{c_1}, \pi_{c_2}, ..., \pi_{c_k}\} \leftarrow instantiateSubSystemPolicies()$; $\mu \leftarrow instantiateMetaPolicy()$
3  $\hat{\sigma}_{c_1}, \hat{\sigma}_{c_2}, ..., \hat{\sigma}_{c_k}, \hat{\sigma}_{\mu} \leftarrow 0$; $\mathscr{L} \leftarrow \{\hat{\sigma}_{c_1}, \hat{\sigma}_{c_2}, ..., \hat{\sigma}_{c_k}\}$; $\mathscr{U} \leftarrow \{\}$; $N_{c_1}, N_{c_2}, ..., N_{c_k} \leftarrow 0$
4  while $\hat{\sigma}_{\mu} \leq 1 - \delta$ do
5      $\{p_{c_1}, p_{c_2}, ..., p_{c_k}\} \leftarrow$ Solution of bilinear program (1)-(6) using $(\tilde{M}, \mathscr{L}, \mathscr{U})$
6      $c_j \leftarrow selectSubSystemToTrain(p_{c_1}, p_{c_2}, ..., p_{c_k}, \hat{\sigma}_{c_1}, \hat{\sigma}_{c_2}, ..., \hat{\sigma}_{c_k})$
7      $\pi_{c_j} \leftarrow RLTrain(c_j, \pi_{c_j}, N_{train})$; $N_{c_j} \leftarrow N_{c_j} + N_{train}$
8      $\hat{\sigma}_{c_j} \leftarrow estimateSubTaskSuccessProbability(c_j, \pi_{c_j})$
9      $\mathscr{L}.update(\hat{\sigma}_{c_j})$
10     if $N_{c_j} \geq N_{max}$ then
11         $\mathscr{U}.add(\hat{\sigma}_{c_j})$
12     $\mu \leftarrow solveOptimalHLMPolicy(\tilde{M}, \mathscr{L})$; $\hat{\sigma}_{\mu} \leftarrow predictTaskSuccessProbability(\tilde{M}, \mu, \mathscr{L})$
13 return $\{\pi_{c_1}, \pi_{c_2}, ..., \pi_{c_k}\}$, $\mu$
```

function, we have $\sigma^c_{\pi_c}(s) = \mathbb{E}[\sum_{t \in [T_c]} R_c(s_t) | \pi_c, s_0 = s]$. We can maximize the probability of reaching an exit condition by maximizing the expected undiscounted sum of rewards over time horizon $T_c$.

To verify that a learned sub-system policy $\pi_c$ satisfies its sub-task specification, we consider $\bar{\sigma}_c = \inf\{\sigma^c_{\pi_c}(s) | s \in \mathscr{I}_c\}$, the greatest lower bound of the policy's probability of sub-task succcess, beginning from any of the sub-system's entry conditions. So long as $\bar{\sigma}_c \geq p_c$, the sub-task specification is satisfied. In practice, the value of $\bar{\sigma}_c$ cannot be known exactly, but we may obtain an estimate $\hat{\sigma}_c$ of its value through empirical rollouts of $\pi_c$, beginning from the different entry conditions $s \in \mathscr{I}_c$. We refer to $\hat{\sigma}_c$ as the *estimated performance value* of policy $\pi_c$.

**Automatic Refinement of the Sub-Task Specifications.** The estimated performance values $\hat{\sigma}_c$ are useful not only for the empirical verification of the learned policies, but also as additional information used periodically during training to refine the sub-task specifications. To do so, we re-solve the optimization problem (1)-(4), with a modified objective (5), and additional constraints (6).

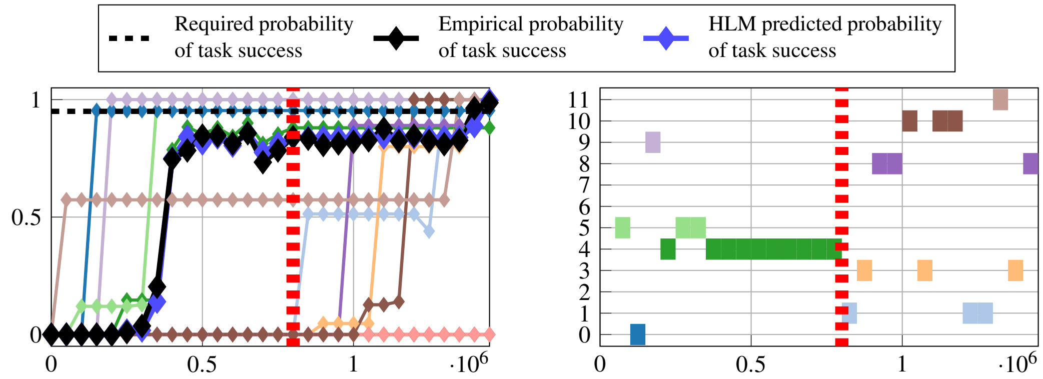
$$obj(\mathscr{L}) = \sum_{c \in \mathscr{C}} (p_c - \hat{\sigma}_c) \tag{5}$$

$$LBConst(\mathscr{L}) = \{p_c \geq \hat{\sigma}_c | \forall \hat{\sigma}_c \in \mathscr{L}\}, \qquad UBConst(\mathscr{U}) = \{p_c \leq \hat{\sigma}_c | \forall \hat{\sigma}_c \in \mathscr{U}\} \tag{6}$$

Here, we assume that the sub-systems have learned policies $\pi_{c_1}, \pi_{c_2}, ..., \pi_{c_k}$. Let $\mathscr{L} = \{\hat{\sigma}_{c_1}, \hat{\sigma}_{c_2}, ..., \hat{\sigma}_{c_k}\}$ be the set of the corresponding estimated performance values. The objective function (5) minimizes the performance gap between the sub-task specifications $p_c$ and the current estimated performance values $\hat{\sigma}_c$. The rationale behind the additional constraints defined by $LBConst(\mathscr{L})$ is as follows: the sub-systems have already learned policies achieving probabilities of sub-task success greater than the estimated performance values $\hat{\sigma}_c$, and so there is no reason to consider sub-task specifications $p_c$ that are below these values.

Conversely, if the RL algorithm of a particular sub-system $c$ has *converged* – i.e. the value of $\hat{\sigma}_c$ will no longer increase with additional training steps – we add the constraint $p_c \leq \hat{\sigma}_c$. This ensures that solutions to the optimization problem will *not* yield a sub-task specification $p_c$ that is larger than what the sub-system can realistically achieve. In practice, as a proxy to convergence, we allow each sub-system a maximum budget of $N_{max}$ training steps. Once any sub-system $c$ has exceeded this training budget, we append $\hat{\sigma}_c$ to the set $\mathscr{U}$, which is used to define $UBConst(\mathscr{U})$ in (6).

**Iterative Compositional Reinforcement Learning (ICRL).** By alternating between the training of the sub-systems, and the refinement of the sub-task specifications, we obtain Algorithm 1. In line 12, the HLM $\tilde{M}$ and the current estimated performance values $\mathscr{L}$ are used to plan a meta-policy $\mu$ maximizing the probability $\hat{\sigma}_{\mu}$ of reaching the HLM goal state $\tilde{s}_{\checkmark}$. These steps use standard MDP solution techniques. The condition in line 4 ensures that the while loop only ends once a meta-policy exists, using the learned sub-system policies, that satisfies the task specification. In line 5, the bilinear program (1)-(6) is solved to update the values of $p_c$. These values are used, along with the estimated performance values, to select a sub-system to train. A simple selection scheme, is to



(a) Estimated task and sub-task success probabilities during training. The y-axis shows probability values and the x-axis shows the total elapsed training steps.

(b) Automatically generated sub-system training schedule. The y-axis shows the sub-system indexes and the x-axis shows the total elapsed training steps.

Figure 2: Experimental results. Each sub-task is represented by a different color, matching to the colors used in Figure 1a. The vertical dotted red lines in (a) and (b) illustrate the point in training at which the HLM automatically refines the sub-task specifications, resulting in a distinct change in the sub-systems that are trained, and in the composite system's resulting behavior.

choose the sub-system $c_j$ maximizing the current performance gap between $p_{c_j}$ and $\hat{\sigma}_{c_j}$. Finally, in line 7, the sub-system is trained for $N_{train}$ steps using the RL algorithm of choice.

## 5 Numerical Examples

**Example Setup.** In this section, we present the results of implementing the proposed framework for the labyrinth navigation task used as a running example throughout the paper. Recall that the overall task specification is to safely navigate from the labyrinth's initial state in the top left corner to the goal state marked by a green square in the bottom left corner, with a probability of at least 0.95. Figure 1a illustrates the labyrinth environment, and highlights each sub-task with a different color, for easier comparison with the results plotted in Figures 2a and 2b.

**Implementation Details.** We implement the labyrinth environment using MiniGrid [4]. The environment's state space consists of the current position and orientation within the labyrinth, and the allowed actions are: *turn left*, *turn right*, and *move forward*. A slip probability is added to the environment dynamics to render them stochastic; each action has a 10% probability of failing and instead causing one of the results of the other actions to occur. Each RL sub-system is trained using the Stable-Baselines3 [22] implementation of the proximal policy optimization (PPO) algorithm [24]. Whenever estimates of task or sub-task success probabilities are needed, we roll out the corresponding (sub-)system 300 times from an initial state, and compute the empirical success rate. We solve the bilinear program in (1)–(4) using Gurobi 9.1 [8]. Gurobi transforms the bilinear program into an equivalent mixed-integer linear program, and computes a globally optimal solution to this program by using cutting plane and branch and bound methods. For further details on the implementation please refer to the supplementary material. Project code is publicly available at: https://github.com/cyrusneary/verifiable-compositional-rl.

**Empirical Validation of Theorem 1.** At regular intervals during training, marked by diamonds in Figure 2a, each sub-system's probability of sub-task success is estimated and used to update $\mathscr{L}$ and $\mathscr{U}$, as described in §4. That is, each diamond in Figure 2a corresponds to a pass through the main loop of algorithm 1. The HLM-predicted probability of the meta-policy completing the overall task is illustrated in Figure 2a by the navy blue curve. For comparison, we plot empirical measurements of the success rate of the meta-policy in black. We clearly observe that the HLM predictions closely match the empirical measurements. This provides an empirical confirmation of the result of Theorem 1, and validates the use of the HLM for verification of the task specification.

**Automatic Sub-Task Specification Refinement Leads to Meta-Policy Adaptation and Targeted Sub-System Training.** Figure 2b illustrates the sub-system training schedule. Table 1 lists the values of $p_c$ for each sub-system $c$. We observe from Table 1 that prior to $8e5$ elapsed training steps, the value of $p_c$ is only specified to be close to 1.0 for sub-systems $c_0$, $c_4$, $c_5$, and, $c_9$. As can be seen in Figure 1a, these are the sub-systems needed to move straight down, through the rooms containing

| Sub-System Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_c$ at $t = 6e5$ | .97 | .00 | .00 | .00 | .97 | 1.0 | .00 | .00 | .00 | 1.0 | .00 | .57 |
| $p_c$ at $t = 10e5$ | .95 | .99 | .00 | .99 | .88 | 1.0 | .00 | .00 | .99 | 1.0 | .99 | .99 |

Table 1: Demonstration of automatic sub-task specification refinement. Each value corresponds to a sub-task specification, i.e. the minimum allowable probability of sub-task success. The two rows of the table show these values at two distinct points of the system's training; before and after the sub-task specification refinement illustrated by the dotted red lines in Figures 2a and 2b. The cells highlighted in grey indicate which sub-systems are used by the meta-policy, at the specified point.

lava, to the goal. The HLM has selected a meta-policy that will only use these sub-systems because their composition yields the shortest path to goal; this path only requires training of 4 of the sub-systems. Furthermore, because the meta-policy does not use any of the other sub-systems, it places no requirements on their probability of sub-task success. Figure 2b agrees with this observation: only this small collection of the sub-systems are trained prior to $8e5$ elapsed training steps. In particular, sub-system 4, which must navigate the top lava room and is represented by dark green, is trained extensively. However, due to the environment slip probability, this sub-system is unable to meet its sub-task specification, *safely navigate to the room's exit with probability 0.97*, regardless of the number of training iterations it receives.

As a result, sub-system 4 exhausts its individual training budget after $8e5$ elapsed system training steps, marked by the vertical dotted red lines in Figures 2a and 2b. At this point, sub-system 4's empirically estimated success rate of 0.88 is used to update the HLM, which then refines the sub-task specifications as described in §4. The result of this refinement is a new meta-policy, which instead uses sub-systems $c_1$, $c_3$, $c_8$, $c_{10}$, and $c_{11}$ to take an alternate path that avoids the lava rooms altogether. The updated sub-task specifications are listed in the second row of Table 1, and in Figure 2b we observe a distinct change in the sub-systems that are trained. Once sub-systems $c_1$, $c_3$, $c_8$, $c_{10}$, and $c_{11}$ learn to satisfy their new sub-task specifications with the requires probability, the composite system's probability of task success rises above 0.95, satisfying the overall task specification.

**Comparison to a Monolithic RL Approach.** The presented numerical results additionally demonstrate the efficiency of the compositional approach to training. By comparison, a monolithic approach in which the entire task is treated as a single sub-system, using the PPO algorithm with the same parameters, takes roughly thirty million training steps to learn a behavior that satisfies the task specification. The proposed ICRL algorithm takes less than two million training steps. While this is not a fair comparison because the proposed compositional approach has access to more prior information, given in the form of the sub-system entry and exit conditions, such information is often available through natural decompositions of complex systems. The proposed framework provides a method to take advantage of such composite systems to great effect.

**Summary of Experimental Findings.** The system initially plans to take the shortest path, through the lava rooms, to reach the goal. However, the chance of failure posed by the lava prevents the system from satisfying its task specification, even after extensive learning. So, the HLM automatically plans an alternate meta-policy that avoids the lava rooms and has a higher probability of successfully reaching the goal, after a certain number of training steps has elapsed. The required probability of satisfying the sub-task specifications are updated accordingly, and the corresponding sub-systems are trained to satisfy them. The resulting composite system behavior is empirically shown to satisfy the task specification. We additionally note that sub-systems $c_2$, $c_6$, and $c_7$, which are not particularly useful in reaching the goal, are not trained at all.

## 6 Conclusions

The verification of reinforcement learning (RL) systems is a critical step towards their widespread deployment in engineering applications. We develop a framework for verifiable and compositional RL in which collections of RL sub-systems are composed to achieve an overall task. Using the framework, we present methods to automatically decompose system-level task specifications into individual sub-task specifications, and to iteratively refine these sub-task specifications while training the sub-systems to satisfy them. Future directions will aim at addressing extensions of the current framework to compositional multi-agent RL systems and to systems involving partial information.

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# Verifiable and Compositional Reinforcement Learning Systems: Supplementary Material

## A Proof of Theorem 1

In this section, we provide a proof of Theorem 1.

**Intuition of the Proof.** While the details of the proof are provided in the remainder of this section, we begin by outlining the intuition behind the proof, which is relatively straightforward. We want to show that if for every sub-system $c \in \mathscr{C}$ and every entry condition $s \in \mathscr{I}_c$ we have $\mathbb{P}^{s}_{M,\pi_c}(\Diamond_{\leq T_c}\mathscr{F}_c) \geq p_c$, then $\mathbb{P}^{s_I}_{M,\mu}(\Diamond\mathscr{F}_{targ}) \geq \mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond\tilde{s}_\checkmark)$. We begin by defining $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond\tilde{s}_\checkmark)$ as the sum of the probabilities of every sequence $\tilde{s}_0c_0\tilde{s}_1c_1...\tilde{s}_m$ of HLM states that eventually reaches the goal state $\tilde{s}_\checkmark$. Then, using the theorem's assumption, we observe that for each such sequence there exists a collection of sequences $s_0c_0a_0s_1c_1a_1...s_n$ of environment states, sub-systems, and actions that has a higher probability value under the measure $\mathbb{P}^{s_I}_{M,\mu}(\cdot)$ than the original sequence had under measure $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\cdot)$. Finally, we note that every such sequence $s_0c_0a_0s_1c_1a_1...s_n$ of environment states, sub-systems, and actions eventually reaches the target set $\mathscr{F}_{targ}$ and that the aforementioned collections of these sequences are pairwise disjoint. From this, we are able to conclude the result of the theorem.

**Preliminary Definitions.** We begin by defining $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond\tilde{s}_\checkmark)$. To do so, we define the probability space associated with the high-level model (HLM) $\tilde{M}$ by following the formalisms laid out in Chapter 10 of [1]. Recall that the HLM $\tilde{M} = (\tilde{S}, \tilde{s}_I, \tilde{s}_\checkmark, \tilde{s}_\times, \mathscr{C}, \tilde{P})$ is a parametric Markov decision process (pMDP) with states $\tilde{S}$, action set $\mathscr{C}$, and transition probability function $\tilde{P}$ parametrized by $p_c$ for $c \in \mathscr{C}$. Also, $\tilde{\mu} : S \times \mathscr{C} \to [0,1]$ is a stationary policy on $\tilde{M}$.

Let $Paths(\tilde{M}, \tilde{\mu}, \tilde{s}_I)$ denote the set of all possible infinite sequences $\tilde{s}_0c_0\tilde{s}_1c_1.. \in (\tilde{S}\times\mathscr{C})^\omega$ such that $\tilde{s}_0 = \tilde{s}_I$ and for every $i \in \{0,1,...\}$, $\tilde{\mu}(\tilde{s}_i, c_i) > 0$ and $\tilde{P}(\tilde{s}_i, c_i, \tilde{s}_{i+1}) > 0$. These are the *infinite paths* of state-action pairs that could occur in MDP $\tilde{M}$ under policy $\tilde{\mu}$, which play the role of the outcomes of our probability space. Then, define $Paths_{fin}(\tilde{M}, \tilde{\mu}, \tilde{s}_I)$ to be the set of all *finite path fragments*, i.e. of all finite sequences $\tilde{s}_0c_0\tilde{s}_1c_1...\tilde{s}_{m-1}c_{m-1}\tilde{s}_m \in (\tilde{s}\times\mathscr{C})^*$ such that $\tilde{s}_0 = \tilde{s}_I$ and for every $i \in \{0,1,...,m-1\}$, $\tilde{\mu}(\tilde{s}_i, c_i) > 0$ and $\tilde{P}(\tilde{s}_i, c_i, \tilde{s}_{i+1}) > 0$. The $\sigma$-algebra – denoted by $\Sigma^{s_I}_{\tilde{M},\tilde{\mu}}$ – which plays the role of the event set of our probability space, is the smallest $\sigma$-algebra containing all *cylinder sets*, denoted $Cyl(\tilde{s}_0c_0...\tilde{s}_m)$, of all finite path fragments $\tilde{s}_0c_0...\tilde{s}_m \in Paths_{fin}(\tilde{M}, \tilde{\mu}, \tilde{s}_I)$. The unique probability measure on the $\sigma$-algebra $\Sigma^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}$ is denoted by $\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}} : \Sigma^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}} \to [0,1]$ and is defined in Chapter 10 of [1].

We now define $\Gamma^{\tilde{s},\tilde{s}_\checkmark}_{\tilde{M},\tilde{\mu}}$ to be the set of all finite path fragments that reach the goal state $\tilde{s}_\checkmark \in \tilde{S}$. That is,

$$\Gamma^{\tilde{s},\tilde{s}_\checkmark}_{\tilde{M},\tilde{\mu}} := Paths_{fin}(\tilde{M}, \tilde{\mu}, \tilde{s}) \cap ((\tilde{S}\setminus\tilde{s}_\checkmark)\times\mathscr{C})^*\{\tilde{s}_\checkmark\}.$$

Then, since the cylinder sets of all these finite path fragments are pairwise disjoint, we have

$$\mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(\Diamond\tilde{s}_\checkmark) = \mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(Cyl(\Gamma^{\tilde{s}_I,\tilde{s}_\checkmark}_{\tilde{M},\tilde{\mu}})) \tag{7}$$

$$= \sum_{\tilde{s}_0c_0...\tilde{s}_m \in \Gamma^{\tilde{s}_I,\tilde{s}_\checkmark}_{\tilde{M},\tilde{\mu}}} \mathbb{P}^{\tilde{s}_I}_{\tilde{M},\tilde{\mu}}(Cyl(\tilde{s}_0c_0...\tilde{s}_m)) \tag{8}$$

In order to define the probability of task success $\mathbb{P}^{s_I}_{M,\mu}(\Diamond\mathscr{F}_{targ})$ in MDP $M$ under meta-policy $\mu$, we may similarly define the infinite paths $Paths(M, \mu, s_I)$ to be the set of all infinite sequences $s_0c_0a_0s_1c_1a_1... \in (S\times\mathscr{C}\times A)^\omega$ that have positive probability under meta-policy $\mu : S\times\mathscr{C} \to [0,1]$ acting in MDP $M$ from initial state $s_I$. By similarly defining the finite path fragments $Paths_{fin}(M, \mu, s_I)$ we may define the $\sigma$-algebra $\Sigma^{s_I}_{M,\mu}$ and the corresponding probability measure $\mathbb{P}^{\tilde{s}_I}_{M,\mu}(\cdot)$.

We now define the set $\Gamma_{M,\mu}^{\tilde{s}_I,\mathscr{F}_{targ}}$ of all finite path fragments $s_0c_0a_0...s_{n-1}c_{n-1}a_{n-1}s_n \in Paths_{fin}(M,\mu,s_I)$ such that for every $t < n$, $s_t \in \mathscr{F}_{c_{t-1}}$ and $\mathscr{F}_{c_{t-1}} = \mathscr{F}_{targ}$ are not both true, and at time $t = n$, $s_n \in \mathscr{F}_{c_{n-1}}$ and $\mathscr{F}_{c_{n-1}} = \mathscr{F}_{targ}$. Then, similarly to as in (7), we have

$$\mathbb{P}_{M,\mu}^{s_I}(\Diamond \mathscr{F}_{targ}) = \mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}^{s_I,\mathscr{F}_{targ}})). \tag{9}$$

Given a finite HLM path fragment $\tilde{s}_0\hat{c}_0...\tilde{s}_m \in Paths_{fin}(\tilde{M},\tilde{\mu},\tilde{s}_I)$, we define the collection of *compatible environment path fragments* $\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_m)$ to be the set of all finite path fragments $s_0c_0a_0...s_n \in Paths_{fin}(M,\mu,s_I)$ such that there exists a collection of *meta-decision times* $0 = \tau_0 < \tau_1 < ... < \tau_m = n$ with $c_{\tau_i} = c_{\tau_i+1} = ... = c_{\tau_{i+1}-1} = \hat{c}_i$, and $s_{\tau_i}, s_{\tau_i+1}, ..., s_{\tau_{i+1}-1} \notin \tilde{s}_{i+1}$, and $s_{\tau_i} \in \tilde{s}_i$ for every $i \in \{0,1,...,m\}$ (recall that HLM states $\tilde{s}$ correspond to collections of environment states $s$). Here we use the hat $\hat{c}_i$ to distinguish the $i^{th}$ sub-system in the HLM path fragment $\tilde{s}_0\hat{c}_0...\tilde{s}_m$ from the sub-system $c_i$ of the same index within the environment path fragment $s_0c_0a_0...s_n$.

**Lemma 1.** *Given any finite path fragment $\tilde{s}_0c_0...\tilde{s}_m \in Paths_{fin}(\tilde{M},\tilde{\mu},\tilde{s}_I)$ such that $\tilde{s}_0,\tilde{s}_1,...,\tilde{s}_m \neq \tilde{s}_\times$. If, for every sub-system $c \in \mathscr{C}$ and for every entry condition $s \in \mathscr{I}_c$ we have $\mathbb{P}_{M,\pi_c}^{s}(\Diamond_{\leq T_c}\mathscr{F}_c) \geq p_c$, then the following inequality holds.*

$$\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_m))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0c_0...\tilde{s}_m)) \tag{10}$$

*Proof.* This inequality follows from the Theorem's assumption that for every sub-system $c \in \mathscr{C}$ and for every entry condition $s \in \mathscr{I}_c$, we have $\sigma_{\pi_c}^{c}(s) \geq p_c$ (recall that $\sigma_{\pi_c}^{c}(s) := \mathbb{P}_{M,\pi_c}^{s}(\Diamond_{\leq T_c}\mathscr{F}_c)$). Given the finite path fragment $\tilde{s}_0c_0...\tilde{s}_m \in Paths_{fin}(\tilde{M},\tilde{\mu},\tilde{s}_I)$ we proceed by induction.

Consider the trivial prefix $\tilde{s}_0 \in Paths_{fin}(\tilde{M},\mu,\tilde{s}_I)$ of the path fragment. By definition, we have $\Gamma_{M,\mu}(\tilde{s}_0) = \{s_0 \in Paths_{fin}(M,\mu,s_I) | s_0 \in \tilde{s}_0\}$. Now, by the definitions of $Paths_{fin}(\tilde{M},\tilde{\mu},\tilde{s}_I)$ and $Paths_{fin}(M,\mu,s_I)$, we have $\tilde{s}_0 = \tilde{s}_I$ and $s_0 = s_I$. This implies that $Cyl(s_0) = Paths(M,\mu,s_I)$ and $Cyl(\tilde{s}_0) = Paths(\tilde{M},\tilde{\mu},\tilde{s}_I)$ and thus, trivially, $\mathbb{P}_{M,\mu}^{s_I}(Cyl(s_0)) = \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0)) = 1$.

Now, for any $0 \leq l \leq m-1$ we consider the prefix $\tilde{s}_0c_0...\tilde{s}_l \in Paths_{fin}(\tilde{M},\tilde{\mu},\tilde{s}_I)$ of the path fragment $\tilde{s}_0c_0...\tilde{s}_lc_l...\tilde{s}_m$. Suppose that $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_l))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0c_0...\tilde{s}_l))$. We may write the probability of $Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_lc_l\tilde{s}_{l+1})))$, corresponding to the prefix of length $l+1$, in terms of the probability of the prefix of length $l$ and the probability of all environment path fragments compatible with the HLM transition from $\tilde{s}_l$ to $\tilde{s}_{l+1}$, as follows: $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_lc_l\tilde{s}_{l+1}))) = \mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_l)))\sum_{s\in\tilde{s}_l}\alpha(s) * \mu(s,c_l) * \mathbb{P}_{M,\pi_{c_l}}^{s}(\Diamond_{\leq T_{c_l}}\tilde{s}_{l+1})$. Here, $\alpha(s)$ is *some* distribution such that $\sum_{s\in\tilde{s}_l}\alpha(s) = 1$. Note that from our definition of the meta-policy $\mu$ in terms of $\tilde{\mu}$, $\mu(s,c) := \tilde{\mu}([s]_R,c)$, we have $\mu(s,c_l) = \tilde{\mu}(\tilde{s}_l,c_l)$ for every $s \in \tilde{s}_l$. Furthermore, as $\tilde{s}_{l+1} \neq \tilde{s}_\times$ by assumption, it must be the case that $\tilde{s}_{l+1} = succ(c_l)$. This implies, by definition, that $\mathscr{F}_{c_l} \subseteq \tilde{s}_{l+1}$. Thus, $\mathbb{P}_{M,\pi_{c_l}}^{s}(\Diamond_{\leq T_{c_l}}\tilde{s}_{l+1}) \geq \mathbb{P}_{M,\pi_{c_l}}^{s}(\Diamond_{\leq T_{c_l}}\mathscr{F}_{c_l}) \geq p_{c_l}$, where the final inequality follows from the lemma's assumption and from the fact that $s \in \mathscr{I}_{c_l}$ for every $s \in \tilde{s}_l$. Putting all this together, we have $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_lc_l\tilde{s}_{l+1}))) \geq \mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_l))) * \tilde{\mu}(\tilde{s}_l,c_l) * p_{c_l}$ and given the assumption of our induction step that $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_l))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0c_0...\tilde{s}_l))$, we obtain the inequality $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0c_0...\tilde{s}_lc_l\tilde{s}_{l+1}))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0c_0...\tilde{s}_l)) * \tilde{\mu}(\tilde{s}_l,c_l) * p_{c_l} = \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0c_0...\tilde{s}_lc_l\tilde{s}_{l+1}))$. Here, the final equality follows from the definition of the transition probability $\tilde{P}(\tilde{s}_l,c_l,\tilde{s}_{l+1})$ in terms of the parameter value $p_{c_l}$. By induction, we conclude the proof.

□

With this lemma in place, we are now ready to prove Theorem 1.

**Theorem 1.** *Let $\mathscr{C} = \{c_1,c_2,...,c_k\}$ be a collection of composable sub-systems with respect to initial state $s_I$ and target set $\mathscr{F}_{targ}$ within the environment MDP $M$. Define $\tilde{M}$ to be the corresponding HLM and let $\tilde{\mu}$ be a policy in $\tilde{M}$. If, for every sub-system $c \in \mathscr{C}$ and for every entry condition $s \in \mathscr{I}_c$, $\sigma_{\pi_c}^{c}(s) \geq p_c$, then $\mathbb{P}_{M,\mu}^{s_I}(\Diamond \mathscr{F}_{targ}) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(\Diamond \tilde{s}_\checkmark)$.*

*Proof.* Consider any finite path fragment $\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}$ that reaches the goal state $\tilde{s}_\checkmark$ in the HLM $\tilde{M}$. By Lemma 1, we know that $\mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0 c_0 ... \tilde{s}_m))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0 c_0 ... \tilde{s}_m))$, which implies that

$$\sum_{\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}} \mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0 c_0 ... \tilde{s}_m))) \geq \sum_{\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}} \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(Cyl(\tilde{s}_0 c_0 ... \tilde{s}_m)) \tag{11}$$

$$= \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(\Diamond \tilde{s}_\checkmark). \tag{12}$$

We note that for any such path fragment $\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}$, if follows from the definition of the HLM goal state $\tilde{s}_m = \tilde{s}_\checkmark$ that $\Gamma_{M,\mu}(\tilde{s}_0 c_0 ... \tilde{s}_m) \subseteq \Gamma_{M,\mu}^{s_I, \mathscr{F}_{targ}}$. Furthermore, the cylinder sets of the compatible environment path fragments $Cyl(\Gamma_{M,\mu}(\tilde{s}_0 c_0 ... \tilde{s}_m))$ for every HLM path fragment $\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}$ are pairwise disjoint. So, we conclude that

$$\mathbb{P}_{M,\mu}^{s_I}(\Diamond \mathscr{F}_{targ}) \geq \sum_{\tilde{s}_0 c_0 ... \tilde{s}_m \in \Gamma_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I, \tilde{s}_\checkmark}} \mathbb{P}_{M,\mu}^{s_I}(Cyl(\Gamma_{M,\mu}(\tilde{s}_0 c_0 ... \tilde{s}_m))) \geq \mathbb{P}_{\tilde{M},\tilde{\mu}}^{\tilde{s}_I}(\Diamond \tilde{s}_\checkmark). \tag{13}$$

□

## B Further Experimental Details

In this section we present details surrounding the training of the individual RL sub-systems, as well as the results of training the composite system multiple times with different random seeds.

**Training the RL Sub-Systems.** Each RL sub-system is trained to reach its exit conditions, given it begins in one of its entry conditions. In the labyrinth experiment, these entry and exit conditions correspond to environment states – locations and orientations within the labyrinth gridworld. As described in §4, the reward signals used to train the sub-systems simply return 1.0 when the corresponding exit state(s) have been reached and 0.0 otherwise. Once an exit condition is reached, the episode ends; it is thus impossible for the sub-systems to receive more than a reward of 1.0 per episode of training. If the sub-system instead collides with any of the lava, then the episode ends and the sub-system has no chance at receiving reward.

To train each RL sub-system, we used the Stable-Baselines3 [22] implementation of the proximal policy optimization (PPO) algorithm [24] with default parameters. The values of these algorithm parameters are listed in Table 2.

During each loop of Algorithm 1, a particular sub-system $c$ is selected to train. The selected sub-system is then trained using the PPO algorithm for $N_{train} = 50,000$ training steps. After its training, a new estimate $\hat{\sigma}_c$ of its probability of sub-task success is obtained by rolling out the sub-system's learned policy 300 separate times from its entry state and counting the number of times the sub-system successfully reaches its exit condition within its allowed time horizon. Each sub-system is given a maximum allowable training budget of $N_{max} = 500,000$ training steps before its most recent estimated performance value $\hat{\sigma}_c$ is added as an upper bound constraint on the corresponding parameter value $p_c$ in the bilinear program (1)-(6) (as described in lines 10-11 in Algorithm 1).

**Hardware Resources Used.** All experiments were run locally on a desktop computer with an Intel i9-9900 3.1 GHz CPU with 32 GB of RAM. A complete training run for the entire composite

| | | | | | |
|---|---|---|---|---|---|
| Learning rate | 2.5e-4 | Steps per update | 512 | Minibatch size | 64 |
| Number of epochs when optimizing surrogate loss | 10 | Discount factor ($\gamma$) | 0.99 | GAE parameter ($\lambda$) | 0.95 |
| Clipping parameter | 0.2 | Value function coefficient | 0.5 | Max gradient norm | 0.5 |

Table 2: PPO algorithm parameter values used for the training of the RL sub-systems.

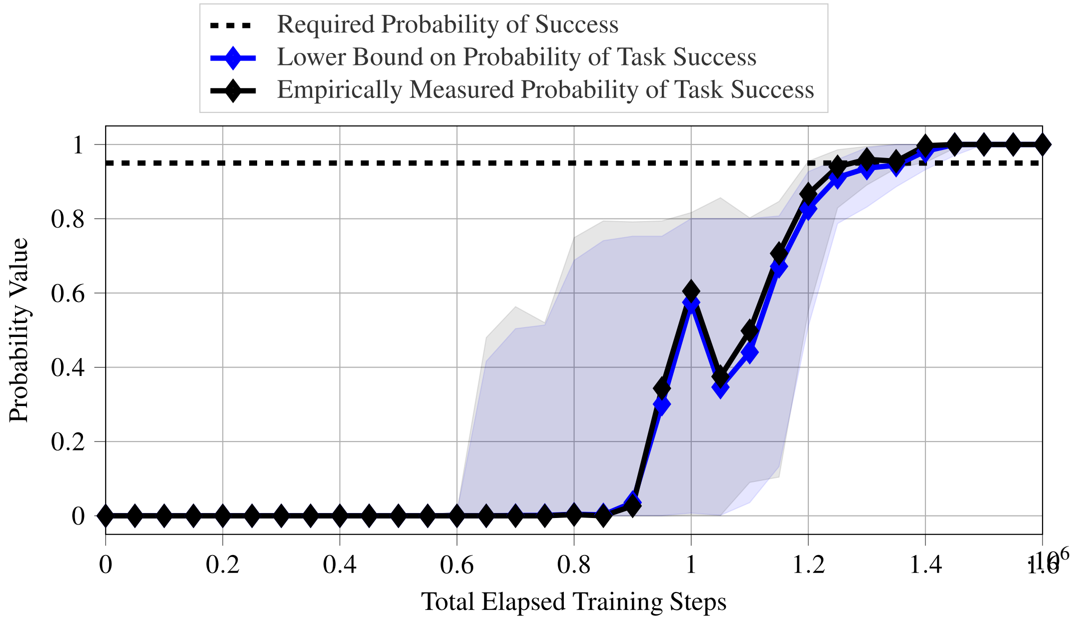

Figure 3: Results of running the labyrinth experiment with 10 different random seeds. The HLM-predicted probability of task success is plotted in blue, while empirical estimates of the system's probability of task success are plotted in black. The solid line visualizes the median values across all runs, while the borders of the shaded regions visualize the $25^{th}$ and $75^{th}$ percentiles.

system, which consists of roughly 1,500,000 total training iterations across all sub-systems, takes approximately 25 minutes of wall-clock time.

**Results of Running Experiment with Different Random Seeds.** To assess the variance in the result of training the system using Algorithm 1, we ran the presented labyrinth experiment 10 separate times with different random seeds. Figure 3 visualizes the results. To avoid unnecessary visual clutter, we did not include information pertaining to the individual sub-systems in this figure. The solid lines indicate the median values across all training runs, while the borders of the shaded region indicate the $25^{th}$ and $75^{th}$ percentiles. We observe high variability in the system's probaility of task success between $0.6e6$ and $1.2e6$ total elapsed training steps. However, we note that by $1.4e6$ training steps, all runs converge to system behavior that satisfies the overall task specification. The variance observe during training is mostly due to the sub-system 4, illustrated by dark green in Figure 1a, that is tasked with navigating the top lava room. In some instances, after roughly $0.6e6$ total training steps, the sub-system learns to navigate past the lava with probability of roughly 0.8 and the entire system's performance rises accordingly. These instances correspond to the top of the shaded region. Conversely, in some instances sub-system 4 doesn't learn to navigate past the lava within its allowed training budget with any probability of success; these instances correspond to the bottom of the shaded region. In both cases, sub-system 4 eventually exhausts its training budget without learning to satisfy its sub-task specification, in which case alternate sub-systems are trained, causing the system's probability of task success to rise above the required threshold after about $1.4e6$ total training iterations.