

# Learning-Based, Safety-Constrained Control from Scarce Data via Reciprocal Barriers

Christos K. Verginis, Franck Djeumou, and Ufuk Topcu

**Abstract**—We develop a control algorithm for the safety of a control-affine system with *unknown* nonlinear dynamics in the sense of confinement in a given safe set. The algorithm leverages robust nonlinear feedback control laws integrated with on-the-fly, data-driven approximations to output a control signal that guarantees the boundedness of the closed-loop system in the given set. More specifically, it first computes estimates of the dynamics based on differential inclusions constructed from data obtained online from a single finite-horizon trajectory. It then computes a novel feedback safety control law that renders the system forward invariant with respect to the safe set, given an accurate enough estimate, using reciprocal barriers. An extension of the algorithm is capable of coping with the controllability loss incurred by the control matrix along the safe set. The algorithm removes a series of common and limiting assumptions considered in the related literature since it does not require global boundedness, growth conditions, or a priori approximations of the unknown dynamics’ terms.

## I. INTRODUCTION

Learning for dynamics and control purposes is an important emerging field concerning autonomous systems. Such systems must be rendered adaptable and robust to unpredictable failures and abrupt changes in the dynamics, necessitating the use of data to synthesize control actions instead of the standard, possibly conservative, model-based design. Moreover, data obtained from the system trajectories before the aforementioned change in the dynamics are unusable and typical episodic reinforcement learning algorithms do not apply; one can only employ limited data obtained on the fly from the current trajectory [1], [2]. Desirable properties of such systems include target stabilization, tracking of a reference trajectory, or safety guarantees [1]–[3].

This paper considers the problem of safety, in the sense of confinement in a given set, of *nonlinear systems* of the form (to be precisely defined in Sec. II)

$$\dot{x} = f(x) + g(x)u \quad (1)$$

with *a priori unknown* terms  $f$  and  $g$ . Unlike previous works in the related literature, we do not impose any of the commonly used assumptions, such as global Lipschitzness, boundedness, [1], [2] or growth conditions [4]. Moreover, we do not assume the commonly used triangular system form or positive definiteness of the control matrix  $g$  [5]–[7], and we do not employ a priori approximations of the system dynamics, such as linear parameterizations [8], [9] or neural networks [10].

We develop a novel two-layered framework for the safety control of the unknown system in (1). We integrate nonlinear

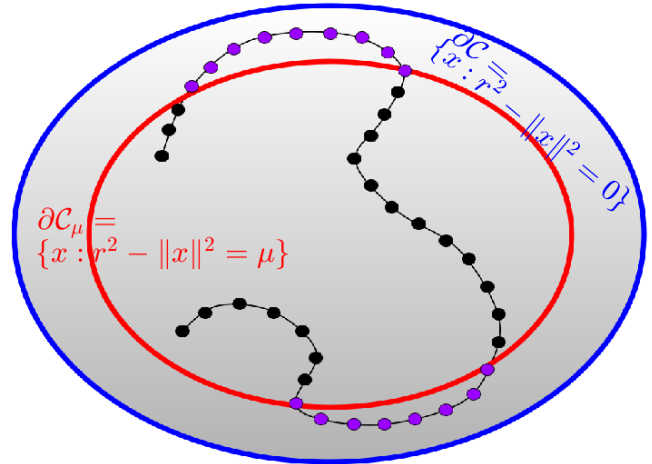


Fig. 1. An example of a safe set  $C = \{x \in \mathbb{R}^n : h(x) = r^2 - \|x\|^2 > 0\}$  (with blue boundary) we aim to retain the system trajectory in, by using data obtained on the fly at the time instants  $\{t_0, t_1, \dots\}$ . The set  $C_\mu = \{x \in \mathbb{R}^n : h(x) = r^2 - \|x\|^2 \in (0, \mu)\}$  (with red boundary) dictates the region where the safety controller is activated.

feedback control with on-the-fly, data-driven techniques to provide an efficient control scheme that guarantees the confinement of the system state in a given set. More specifically, the main contributions are as follows. Firstly, we compute an estimate of the control matrix  $g$ , which is updated on the fly based on a *discrete* set of data from a single finite-horizon trajectory. We then use this estimate to design a novel feedback control protocol based on reciprocal barriers, rendering the system forward-invariant with respect to the given safe set under certain assumptions on the estimation error. The only a priori information required on the dynamics is restricted to upper bounds of Lipschitz constants in the safe set. Secondly, we apply the proposed methodology to “minimally invasive” *local* safety control in the sense that it acts only close to the boundary of the safe set. Finally, we provide an extension that tackles controllability loss incurred by the control matrix  $g$ . The proposed two-layered algorithm does not require any expensive numerical operations or tedious analytic expressions to produce the control signal, enhancing thus its applicability.

Safety of *uncertain* autonomous systems in the sense of set invariance [11] is a topic that has been and still is undergoing intense study by the control community. The works in [4], [7] incorporate uncertainties in safety algorithms based on artificial potential fields, limited, however, to constant unknown parameters and terms that satisfy dissipative or growth conditions. Optimization-based algorithms that

guarantee safety through state constraints [12], [13] cannot tackle dynamic uncertainties more sophisticated than additive bounded disturbances. The recent works in [14], [15] employ adaptive control techniques to tackle safety constraints for uncertain systems, restricted, however, to linearly parameterized dynamics with *constant* unknown parameters. In this paper, we consider a more general class of systems where the functions  $f$  and  $g$  are entirely unknown.

A widely employed methodology that tackles safety for autonomous systems and has received significant attention is the concept of barrier certificates [16], which provide a convenient and efficient way to guarantee invariance in a given set [6], [17]. Nevertheless, standard control based on barrier certificates relies heavily on the underlying dynamics since the respective terms are used in the control design; [8], [9], [18], [19] investigate uncertainties in the system dynamics, restricted, however, to additive perturbations or linearly parameterized terms that include constant unknown parameters. Therefore, the respective methodologies are not applicable to the class of systems considered in this paper.

Another class of work dealing with unknown dynamics is that of funnel control, which guarantees confinement of the state in a given funnel, incorporating transient (e.g., safety) constraints [5], [10], [20], [21]. In contrast to the setup of the current paper, such methodologies either apply to certain forms of the dynamics, such as triangular systems, with possibly positive definite input matrices [5], [21], or rely on approximation of the dynamics using neural networks [10]. The latter has the drawbacks of lacking good heuristics for choosing radial basis functions and number of layers, as well as relying on strong assumptions on the amount of data.

Data-driven approaches have also been integrated with barrier functions to address the safety of uncertain systems through Gaussian-process models and reinforcement learning techniques [3], [22]–[24]. The aforementioned works, however, consider only additive uncertain terms that are assumed to evolve in compact sets. On the contrary, we consider nonlinear systems of the form (1) where both  $f$  and  $g$  are unknown, without having access to any nominal model. Moreover, reinforcement learning and Gaussian-process algorithms typically require large amounts of data in order to provide accurate enough results [2], [22]. Recent methodologies that employ limited data obtained on the fly have been developed in [1], [2], [25], imposing, however, restrictive assumptions on the dynamics, such as global boundedness and Lipschitz continuity with known bounds, or known bounds on the approximation errors. In addition, the aforementioned works resort to online optimization techniques for safety specifications, increasing thus the complexity of the resulting protocols. In the current paper, we rely on limited data without imposing the assumptions stated above, developing a safety control algorithm that does not require significant computational resources.

The remainder of this article is structured as follows. Section II gives the problem formulation. Section III presents the on-the-fly algorithm that approximates of the dynamics, and the control design is provided in Section IV. Section

V investigates the case of controllability loss, and Section VI presents a simulation example. Finally, Section VII concludes the paper.

## II. PROBLEM FORMULATION

### A. Notation

We denote by  $\bar{\mathbb{N}} := \mathbb{N} \cup \{0\}$  the set of nonnegative integer numbers, where  $\mathbb{N}$  is the set of natural numbers. The set of  $n$ -dimensional nonnegative reals, with  $n \in \mathbb{N}$ , is denoted by  $\mathbb{R}_{\geq 0}^n$ ;  $\text{Int}(A)$  and  $\partial A$  denote the interior and boundary, respectively, of a set  $A \subseteq \mathbb{R}^n$ . Given  $a \in \mathbb{R}^n$ ,  $\|a\|$  denotes its 2-norm;  $\nabla h(x) \in \mathbb{R}^n$  is the gradient of a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ . An interval in  $\mathbb{R}$  is denoted by  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$  and the set of intervals on  $\mathbb{R}$  by  $\mathbb{I}\mathbb{R} := \{\underline{A}, \bar{A} : \underline{A}, \bar{A} \in \mathbb{R}, \underline{A} \leq \bar{A}\}$ , which extends to the sets of interval vectors  $\mathbb{I}\mathbb{R}^n$  and matrices  $\mathbb{I}\mathbb{R}^{n \times m}$ . We carry forward the definitions [26] of arithmetic operations, set inclusion, and intersections of intervals to interval vectors and matrices componentwise.

### B. Problem Setup

Consider a system characterized by  $x := [x_1, \dots, x_n]^\top \in \mathbb{R}^n$  with dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u \quad (2)$$

where  $f := [f_1, \dots, f_n]^\top : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g := [g_{ij}] : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are *unknown*, continuously differentiable functions, and  $u := [u_1, \dots, u_m]^\top \in \mathbb{R}^m$  is the control input. The problem this work considers is the invariance of the unknown system (2) in a given closed set  $\mathcal{C} \subset \mathbb{R}^n$  of the form

$$\mathcal{C} := \{x \in \mathbb{R}^n : h(x) \geq 0\} \quad (3a)$$

where  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable function, with bounded derivative  $\frac{dh(x)}{dx}$  in  $\text{Int}(\mathcal{C})$ , and satisfying the simple controllability condition that  $\nabla h(x)^\top g(x)$  is not identically zero (i.e., relative degree one). More specifically, we aim to design a control law that achieves  $x(t) \in \text{Int}(\mathcal{C})$ , i.e.,  $h(x(t)) > 0$ , for all  $t \geq t_0$ , given that  $x(t_0) \in \text{Int}(\mathcal{C})$  for an initial time constant  $t_0 \geq 0$ .

As mentioned in Section I, we aim to integrate a nonlinear feedback control scheme with a data-driven algorithm that approximates the dynamics (2) by using data obtained on the fly from a finite-horizon trajectory. More specifically, consider an increasing time sequence  $\{t_0, t_1, t_2, \dots\}$  signifying the time instants of data measurements. That is, we assume that at each  $t_i$ ,  $i \in \mathbb{N}$ , the system has access to the discrete dataset of  $i$  points  $\mathcal{T}_i := \{(x^j, \dot{x}^j, u^j)\}_{j=0}^{i-1}$ , consisting of the system state  $x^j = [x_1^j, \dots, x_n^j]^\top = x(t_j)$ , the state derivative  $\dot{x}^j = [\dot{x}_1^j, \dots, \dot{x}_n^j]^\top = \dot{x}(t_j)$ , and the control input  $u^j = [u_1^j, \dots, u_m^j]^\top = u(t_j)$  from a trajectory of (2). The trajectory that produces the dataset  $\mathcal{T}_i$  has finite horizon in the sense that, for each finite  $i$ ,  $\mathcal{T}_i$  is finite. We are now ready to give the problem statement treated in this paper.

**Problem 1.** *Let a system evolve subject to the unknown dynamics (2). Given the discrete dataset  $\mathcal{T}_i$ ,  $i \in \mathbb{N}$ , and*

$x(t_0) \in \text{Int}(\mathcal{C})$ , compute a time-varying feedback control law  $u : \mathbb{R}^n \times [t_0, \infty) \rightarrow \mathbb{R}^m$  that guarantees  $x(t) \in \text{Int}(\mathcal{C})$ , for all  $t \geq t_0$ .

We further impose the following assumptions, required for the solution of Problem 1.

**Assumption 1.** *It holds that  $\mathcal{C} \subset \mathcal{B}_r(0)$ , where  $\mathcal{B}_r(0)$  is the open ball of radius  $r$  centered at 0, for some  $r > 0$ .*

**Assumption 2.** *There exist known positive constants  $\bar{f}_k, \bar{g}_{k\ell}$  satisfying  $|f_k(x) - f_k(y)| \leq \bar{f}_k|x - y|$ ,  $|g_{k\ell}(x) - g_{k\ell}(y)| \leq \bar{g}_{k\ell}|x - y|$ , for all  $k \in \{1, \dots, n\}$ ,  $\ell \in \{1, \dots, m\}$ ,  $x, y \in \mathcal{C}$ .*

Assumption 1 simply states that the system remains bounded when in the safe set  $\mathcal{C}$ . Note that we *do not assume* that the system is bounded in any set. Assumption 2 considers knowledge of upper bounds of the Lipschitz constants of  $f_k$  and  $g_{k\ell}$  in the safe set  $\mathcal{C}$ . Note that we do not assume that the functions  $f(\cdot)$  and  $g(\cdot)$  are globally Lipschitz (as e.g., in [1], [2]). The Lipschitz constants of  $f_k$  and  $g_{k\ell}$  in  $\mathcal{C}$  are also not considered to be *exactly* known; rather, upper bounds are needed. In fact, Assumption 2 can be relaxed in the sense that such upper bounds can be computed on the fly using the data from the current system trajectory  $\mathcal{T}_i$  [2].

Note that the current problem setting exhibits a unique challenge due to the on-the-fly availability of the data measurements and the minor assumptions imposed on the dynamics (2). In contrast to most related works, we do not assume global boundedness, Lipschitzness, or growth conditions on the dynamic terms, and we do not employ a priori approximation structures or data obtained offline.

The solution of Problem 1, consisting of a two-layered approach, is given in Sec. III-V. Firstly, we use previous results on on-the-fly approximation of the unknown dynamics [2] and compute locally Lipschitz estimates for  $g(x)$ . Secondly, we use these estimates to design a closed-form feedback control law based on reciprocal barrier functions.

### III. ON-THE-FLY OVER-APPROXIMATION OF THE DYNAMICS

In this section, we provide a brief overview of the approximation algorithm of [2] based on data obtained online from a single finite-horizon trajectory. More specifically, at each  $t_i$ ,  $i \in \mathbb{N}$ , the algorithm uses the information from the finite dataset  $\mathcal{T}_i$  in order to construct a data-driven differential inclusion  $\dot{x} \in \mathcal{F}^i(x) + \mathcal{G}^i(x)u$  that contains the unknown vector field of (2), where  $\mathcal{F}^i : \mathbb{R}^n \rightarrow \mathbb{IR}^n$  and  $\mathcal{G}^i : \mathbb{R}^n \rightarrow \mathbb{IR}^{n \times m}$  are known interval-valued functions. Such an over-approximation enables us to provide a locally Lipschitz estimate  $\hat{g}^i$  of  $g$  to be used in the subsequent feedback control scheme. First, we provide in Lemma 1 closed-form expressions for  $\mathcal{F}$  and  $\mathcal{G}$  given over-approximations of  $f$  and  $g$  at some states.

**Lemma 1** ([2], Lemma 2). *Let  $i \in \mathbb{N}$  and consider the sets  $\mathcal{A} \subseteq \mathcal{C}$ ,  $\mathcal{E}_i := \{(x^j, C_{\mathcal{F}}^j, C_{\mathcal{G}}^j)\}_{j=0}^{i-1}$  where  $C_{\mathcal{F}}^j := (C_{\mathcal{F}_1}^j, \dots, C_{\mathcal{F}_n}^j) \in \mathbb{IR}^n$ ,  $C_{\mathcal{G}}^j := (C_{\mathcal{G}_{k\ell}}^j) \in \mathbb{IR}^{n \times m}$  are intervals satisfying  $f(x^j) \in C_{\mathcal{F}}^j$  and  $g(x^j) \in C_{\mathcal{G}}^j$ . Further,*

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#### Algorithm 1 Approximate( $t_i, \mathcal{T}_i$ )

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**Input:** Single trajectory  $\mathcal{T}_i$ , constant  $M > 0$ , upper bounds on the Lipschitz constants.

**Output:**  $\mathcal{E}_i = \{(x^j, C_{\mathcal{F}}^j, C_{\mathcal{G}}^j) | f(x^j) \in C_{\mathcal{F}}^j, G(x^j) \in C_{\mathcal{G}}^j\}_{j=0}^{i-1}$

- 1:  $\mathcal{A} \leftarrow \mathcal{C}$ ,  $\mathcal{R}^{f_A} \leftarrow [-M, M]^n$ ,  $\mathcal{R}^{G_A} \leftarrow [-M, M]^{n \times m}$
- 2: Define  $x^0 \in \mathcal{A}$ ,  $C_{\mathcal{F}}^0 \leftarrow \mathcal{R}^{f_A}$ , and  $C_{\mathcal{G}}^0 \leftarrow \mathcal{R}^{G_A}$
- 3: **for**  $\iota \in \{1, \dots, i-1\} \wedge (x^\iota, \dot{x}^\iota, u^\iota) \in \mathcal{T}_i$  **do**
- 4:   Compute  $\mathcal{F}^\iota := \mathbf{F}(x^\iota)$ ,  $\mathcal{G}^\iota := \mathbf{G}(x^\iota)$  via (4) using  $\mathcal{E}_{\iota-1}$
- 5:   Compute  $C_{\mathcal{F}}^\iota, C_{\mathcal{G}}^\iota$  via (5),  $\mathcal{F}^\iota, \mathcal{G}^\iota$ , and  $(x^\iota, \dot{x}^\iota, u^\iota)$
- 6: **end for**
- 7: **do**
- 8:   Execute lines 3–6 with  $\mathcal{E}_i$  instead of  $\mathcal{E}_{\iota-1}$  on line 4
- 9: **while**  $\mathcal{E}_N$  is not invariant
- 10: **return**  $\mathcal{E}_i$

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consider the constants  $\bar{f}_k$  and  $\bar{g}_{k\ell}$  satisfying Assumption 2, for all  $k \in \{1, \dots, n\}$ ,  $\ell \in \{1, \dots, m\}$ , and  $x, y \in \mathcal{A}$ . The interval-valued functions  $\mathbf{F} := (\mathbf{F}_1, \dots, \mathbf{F}_n) : \mathbb{R}^n \rightarrow \mathbb{IR}^n$  and  $\mathbf{G} := (\mathbf{G}_{k\ell}) : \mathbb{R}^n \rightarrow \mathbb{IR}^{n \times m}$ , given for all  $k \in \{1, \dots, n\}$  and  $\ell \in \{1, \dots, m\}$ , by the expressions

$$\mathbf{F}_k(x) := \bigcap_{(x^j, C_{\mathcal{F}_k}^j) \in \mathcal{E}_i} \left\{ C_{\mathcal{F}_k}^j + \bar{f}_k \|x - x^j\| [-1, 1] \right\}, \quad (4a)$$

$$\mathbf{G}_{k\ell}(x) := \bigcap_{(x^j, C_{\mathcal{G}_{k\ell}}^j) \in \mathcal{E}_i} \left\{ C_{\mathcal{G}_{k\ell}}^j + \bar{g}_{k\ell} \|x - x^j\| [-1, 1] \right\}, \quad (4b)$$

satisfy  $f(x) \in \mathbf{F}(x)$  and  $g(x) \in \mathbf{G}(x)$ , for all  $x \in \mathcal{A}$ .

Loosely speaking, Lemma 1 states that if a set  $\mathcal{E}_i = \{(x^j, C_{\mathcal{F}}^j, C_{\mathcal{G}}^j)\}_{j=0}^{i-1}$  and Lipschitz bounds are given, it is possible to obtain an analytic formula over the interval domain to over-approximate the unknown  $f$  and  $g$ . Lemma 2 enables to compute the set  $\mathcal{E}_i$  based on the dataset  $\mathcal{T}_i$ .

**Lemma 2** ([2], Lemma 1). *Let a data point  $(x^j, \dot{x}^j, u^j)$ , a vector interval  $\mathcal{F}^j := (\mathcal{F}_1^j, \dots, \mathcal{F}_n^j)^\top \in \mathbb{IR}^n$  such that  $f(x^j) \in \mathcal{F}^j$ , and a matrix interval  $\mathcal{G}^j := (\mathcal{G}_{k\ell}^j) \in \mathbb{IR}^{n \times m}$  such that  $g(x^j) \in \mathcal{G}^j$ . Consider the intervals  $C_{\mathcal{F}}^j := (C_{\mathcal{F}_1}^j, \dots, C_{\mathcal{F}_n}^j) \in \mathbb{IR}^n$  and  $C_{\mathcal{G}}^j := (C_{\mathcal{G}_{k\ell}}^j) \in \mathbb{IR}^{n \times m}$ , defined sequentially for  $\ell \in \{1, \dots, m\}$  by*

$$C_{\mathcal{F}_k}^j := \left\{ \mathcal{F}_k^j \right\} \cap \left\{ \dot{x}_k^j - \mathcal{Y}_k^j \right\}, \quad (5a)$$

$$s_{0,k} := \left\{ \dot{x}_k^j - C_{\mathcal{F}_k}^j \right\} \cap \left\{ \mathcal{Y}_k^j \right\}, \quad (5b)$$

$$C_{\mathcal{G}_{k\ell}}^j := \begin{cases} \left( \left\{ s_{\ell-1,k} - \sum_{p>\ell} \mathcal{G}_{kp}^j u_p^j \right\} \cap \left\{ \mathcal{G}_{k\ell}^j u_\ell^j \right\} \right) \frac{1}{u_\ell^j}, & u_\ell^j \neq 0 \\ \mathcal{G}_{k\ell}^j, & \text{otherwise,} \end{cases} \quad (5c)$$

$$s_{\ell,k} := \left\{ s_{\ell-1,k} - C_{\mathcal{G}_{k\ell}}^j u_\ell^j \right\} \cap \left\{ \sum_{p>\ell} \mathcal{G}_{kp}^j u_p^j \right\}, \quad (5d)$$

for all  $k \in \{1, \dots, n\}$ , where  $\mathcal{Y}^j := \mathcal{G}^j u^j \in \mathbb{IR}^n$ . Then,  $C_{\mathcal{F}}^i$  and  $C_{\mathcal{G}}^i$  are the smallest intervals enclosing  $f(x^j)$  and  $g(x^j)$ , respectively, given only  $(x^j, \dot{x}^j, u^j)$ ,  $\mathcal{F}^j$ , and  $\mathcal{G}^j$ .

Using Lemma 2, Alg. 1 utilizes the dataset  $\mathcal{T}_i$  at each time instant  $t_i$ ,  $i \in \mathbb{N}$ , to compute the set  $\mathcal{E}_i$  and subsequently,

over-approximate  $f$  and  $g$  using (4). Initially,  $C_{\mathcal{F}}^0$  and  $C_{\mathcal{G}}^0$  need to be computed such that  $f(x^0) \in C_{\mathcal{F}}^0$ ,  $g(x^0) \in C_{\mathcal{G}}^0$ , which can be achieved by choosing a sufficiently large  $M$  (see line 2 of Alg. 1). Note that the computational complexity of the algorithm (in time and memory) is linear in the number of the elements of  $\mathcal{T}_i$  and the system dimension  $n$ . The subsequent theorem characterizes the correctness of the obtained differential inclusions.

**Theorem 1** ([2], Theorem 1). *Let  $i \in \mathbb{N}$  and  $F^i := (F_1^i, \dots, F_n^i)^\top : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G^i := (G_{k\ell}^i) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ , with  $F^i(x) := \mathbf{F}(x)$ ,  $G^i(x) := \mathbf{G}(x)$  computed from (4) and the output  $\mathcal{E}_i$  of Alg. 1, which is executed at  $t_i$  using the dataset  $\mathcal{T}_i$ . Then it holds that  $\dot{x}(t) \in F^i(x(t)) + G^i(x(t))u$ , for all  $t \geq t_i$ .*

**Remark 1.** *As pointed out in [2], Alg. 1 can be adjusted to employ extra information on  $f$  and  $g$ , if available, yielding more accurate approximations. In particular, if we are given sets  $\mathcal{A} \subseteq \mathcal{C}$ ,  $\mathcal{R}^{\mathcal{F}, \mathcal{A}}$ ,  $\mathcal{R}^{\mathcal{G}, \mathcal{A}}$  such that  $\{f(x)|x \in \mathcal{A}\} \subseteq \mathcal{R}^{\mathcal{F}, \mathcal{A}}$  and  $\{g(x)|x \in \mathcal{A}\} \subseteq \mathcal{R}^{\mathcal{G}, \mathcal{A}}$ , these can be used in Alg. 1, replacing the respective ones defined in line 1. We stress, nevertheless, that such sets are not required to be available.*

Based on Lemmas 1 and 2, we propose now a locally Lipschitz function  $\hat{g}^i : \mathcal{C} \rightarrow \mathbb{R}^{n \times m}$  that estimates the unknown function  $g$  at each measurement instant  $t_i$ .

**Lemma 3.** *Let  $i \in \mathbb{N}$ . Given a weight  $\theta \in [0, 1]$  and a set  $\mathcal{A} \subseteq \mathcal{C}$ , each component of the function  $\hat{g}^i := [\hat{g}_{k\ell}^i] : \mathcal{C} \rightarrow \mathbb{R}^{n \times m}$ , for all  $x \in \mathcal{A}$ , given by*

$$\hat{g}_{k\ell}^i(x) = \theta \underline{G}_{k\ell}^i(x) + (1 - \theta) \bar{G}_{k\ell}^i(x), \quad (6)$$

where  $\underline{G}_{k\ell}^i$  and  $\bar{G}_{k\ell}^i$  are the left and right endpoints, respectively, of the interval  $G_{k\ell}^i$ , is locally Lipschitz in  $\mathcal{A}$ , for all  $k \in \{1, \dots, n\}$  and  $\ell \in \{1, \dots, m\}$ .

*Proof:* By the definition of  $G_{k\ell}^i$ , given by the output  $\mathcal{E}_i$  of Alg. 1 and Lemma 1, we have for any  $x \in \mathcal{A}$  that

$$\underline{G}_{k\ell}^i(x) = \inf_{(x^j, \cdot, C_{\mathcal{G}_{k\ell}}^j) \in \mathcal{E}_i} \left\{ C_{\mathcal{G}_{k\ell}}^j + \bar{g}_{k\ell} \|x - x^j\|[-1, 1] \right\},$$

$$\bar{G}_{k\ell}^i(x) = \sup_{(x^j, \cdot, C_{\mathcal{G}_{k\ell}}^j) \in \mathcal{E}_i} \left\{ C_{\mathcal{G}_{k\ell}}^j + \bar{g}_{k\ell} \|x - x^j\|[-1, 1] \right\},$$

for all  $k \in \{1, \dots, n\}$  and  $\ell \in \{1, \dots, m\}$ . Thus, one can observe through interval arithmetic that  $\underline{G}_{k\ell}^i(x)$  and  $\bar{G}_{k\ell}^i(x)$  can be also written as

$$\underline{G}_{k\ell}^i(x) = \inf_{(x^j, \cdot, C_{\mathcal{G}_{k\ell}}^j) \in \mathcal{E}_i} \left\{ m_u(x^j) + \bar{g}_{k\ell} \|x - x^j\| \right\},$$

$$\bar{G}_{k\ell}^i(x) = \sup_{(x^j, \cdot, C_{\mathcal{G}_{k\ell}}^j) \in \mathcal{E}_i} \left\{ m_l(x^j) - \bar{g}_{k\ell} \|x - x^j\| \right\},$$

where  $m_l(x^j) := \inf C_{\mathcal{G}_{k\ell}}^j$  and  $m_u(x^j) := \sup C_{\mathcal{G}_{k\ell}}^j$ . Using the inequality  $|||x| - |y||| \leq \|x - y\|$  for any two vectors  $x, y \in \mathbb{R}^n$ , we conclude that the functions  $h^j : x \mapsto m_u(x^j) + \bar{g}_{k\ell} \|x - x^j\|$  and  $l^j : x \mapsto m_l(x^j) - \bar{g}_{k\ell} \|x - x^j\|$  are Lipschitz continuous in  $\mathcal{A}$  with  $\bar{g}_{k\ell}$  as the Lipschitz constant in  $\mathcal{A}$ . Similarly, and by using  $\max\{l, h\} = 0.5(l +$

$h + |l - h|)$ , and  $\min\{l, h\} = 0.5(l + h - |l - h|)$ , we can deduce by induction that  $x \mapsto \sup_{j \in \{1, \dots, i\}} l_j(x)$  and  $x \mapsto \inf_{j \in \{1, \dots, i\}} h_j(x)$  are also Lipschitz continuous owing to the Lipschitz continuity of  $l^j$  and  $h^j$  for all  $j \in \{1, \dots, n\}$ . Thus, one obtains that  $x \mapsto \bar{G}_{k\ell}^i(x)$  and  $x \mapsto \underline{G}_{k\ell}^i(x)$  are also Lipschitz continuous, from which we conclude the Lipschitz continuity of  $\hat{g}_{k\ell}$ , being a linear combination of the latter.

#### IV. CONTROL DESIGN

This section presents a control algorithm based on the approximation of the dynamics of Section III and the concept of reciprocal barriers. Given the set  $\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}$  and following [6], we define the cont. differentiable reciprocal barrier function  $\beta : (0, \infty) \rightarrow \mathbb{R}$  that satisfies

$$\frac{1}{\alpha_1(h)} \leq \beta(h) \leq \frac{1}{\alpha_2(h)} \quad (7)$$

for class  $\mathcal{K}$  functions  $\alpha_1, \alpha_2$ . Note that (7) implies  $\inf_{x \in \text{Int}(\mathcal{C})} \beta(h(x)) > 0$  and  $\lim_{x \rightarrow \partial \mathcal{C}} \beta(h(x)) = \infty$ .

According to [6], the set  $\text{Int}(\mathcal{C})$  is forward-invariant if there exists a reciprocal barrier function  $\beta$  satisfying (7) and

$$\frac{d\beta(h(x))^\top}{dx} (f(x) + g(x)u) \leq \alpha_3(h(x)) \quad (8)$$

for a class  $\mathcal{K}$  function  $\alpha_3$ . This condition allows  $\beta$  to grow quickly when solutions are far away from  $\partial \mathcal{C}$ , with the growth rate approaching zero as solutions approach  $\partial \mathcal{C}$ . For unknown  $f$  and  $g$ , however, the design of a controller  $u$  such that (8) holds, can be difficult, if not impossible, to achieve. Therefore, this work focuses on maintaining  $\beta$  bounded in a compact set, which implies that  $h$  is kept away from zero. To this end, we enforce the extra condition on  $\beta$ :

$$\left\| \frac{d\beta(h)}{dh} \right\| \leq \frac{1}{\alpha_4(h)}, \quad (9)$$

for a class  $\mathcal{K}$  function  $\alpha_4$ , which essentially implies that the derivative of  $\beta(\cdot)$  is bounded in  $\text{Int}(\mathcal{C})$ . Examples of  $\beta$  include  $\beta(x) = \frac{1}{x}$ ,  $\beta(x) = -\ln\left(\frac{x}{1+x}\right)$ .

Let now the estimate  $\hat{g}^i(x)$  of  $g(x)$ , as computed by Lemma 3, for  $t \in [t_i, t_{i+1})$ ,  $i \in \bar{\mathbb{N}}^1$ . Let also the respective error  $\tilde{g}^i(x) := \hat{g}^i(x) - g(x)$  for  $i \in \bar{\mathbb{N}}$ . The term  $\hat{g}^i(x)$  will be used in the control design to cancel the effect of  $g(x)$ , inducing thus a time-dependent switching. The next theorem is the first result of this paper, stating that, if the estimated system defined by  $\hat{g}^i(x)$  is controllable with respect to  $h(x)$ , and if  $\hat{g}^i(x)$  is sufficiently close to  $g(x)$ , we achieve boundedness of  $x(t)$  in  $\text{Int}(\mathcal{C})$  and, consequently, provide a solution to Problem 1.

**Theorem 2.** *Let a system evolve according to (2) and a set  $\mathcal{C}$  satisfying  $x(t_0) \in \text{Int}(\mathcal{C})$  for some  $t_0 \geq 0$ . Let functions  $h : \text{Int}(\mathcal{C}) \rightarrow [0, \infty)$ ,  $\beta : (0, \infty) \rightarrow \mathbb{R}$  satisfying (3) and (7), (9), respectively. Assume that*

$$\|\hat{g}^i(x)^\top \nabla h(x)\| \geq \varepsilon, \quad (10a)$$

$$\bar{g} \bar{h} < \varepsilon, \quad (10b)$$

<sup>1</sup>Since  $\mathcal{T}_0$  is empty,  $\hat{g}^0(x)$  is set randomly.

for a positive constant  $\varepsilon$ , for all  $i \in \bar{\mathbb{N}}$  and  $x \in \text{Int}(\mathcal{C})$ , and where  $\bar{h} := \sup_{x \in \text{Int}(\mathcal{C})} \|\nabla h(x)\|$  and  $\bar{g} := \sup_{x \in \text{Int}(\mathcal{C})} \|\tilde{g}^i(x)\|$ . Under Assumptions 1, 2, the control law

$$u := u(x, t) = -\kappa \frac{d\beta(h(x))}{dh(x)} \frac{\hat{g}^i(x)^\top \nabla h(x)}{\|\hat{g}^i(x)^\top \nabla h(x)\|^2}, \quad (11)$$

for  $t \in [t_i, t_{i+1})$ ,  $i \in \bar{\mathbb{N}}$  and where  $\kappa$  is a positive constant control gain, guarantees that  $x(t) \in \text{Int}(\mathcal{C})$ , and the boundedness of all closed loop signals, for all  $t > t_0$ .

*Proof:* The closed-loop system  $\dot{x} = f(x) + g(x)u(x, t)$  is piecewise continuous in  $t \geq t_0$ , for each fixed  $x \in \text{Int}(\mathcal{C})$ , and, in view of Lemma 3, locally Lipschitz in  $x \in \text{Int}(\mathcal{C})$  for each fixed  $t \geq t_0$ . Hence, since  $x(t_0) \in \text{Int}(\mathcal{C})$ , we conclude from [27, Theorem 2.1.3] the existence of a maximal absolutely continuous solution  $x(t)$ , satisfying  $x(t) \in \text{Int}(\mathcal{C})$ , for all  $t \in [t_0, t_{\max})$ , for a positive constant  $t_{\max} > t_0$ . We aim to prove next that  $x(t)$  remains in a compact subset of  $\text{Int}(\mathcal{C})$  and consequently  $t_{\max} = \infty$ .

Since  $x(t) \in \text{Int}(\mathcal{C})$  for all  $t \in [t_0, t_{\max})$ , Alg. 1 produces valid approximation sets  $F^i(x)$ ,  $G^i(x)$  at each  $t_i$ , from which we obtain the estimate  $\hat{g}^i(x)$ . Using  $g(x) = \hat{g}^i(x) + \tilde{g}^i(x)$  and  $\beta_d := \frac{d\beta(h(x))}{dh(x)}$ , one obtains

$$\dot{\beta} \leq \beta_d \nabla h(x)^\top f(x) - \kappa \beta_d^2 + \kappa \beta_d^2 \frac{\|\tilde{g}^i(x)^\top \nabla h(x)\|}{\|\hat{g}^i(x)^\top \nabla h(x)\|}$$

for all  $t \in [t_0, t_{\max})$ . Since  $x \in \text{Int}(\mathcal{C})$  and  $f$  is cont. differentiable, there exists a constant  $\bar{f}$ , independent of  $t_{\max}$ , satisfying  $\|f(x(t))\| \leq \bar{f}$ , for all  $t \in [t_0, t_{\max})$ . Moreover, since  $\|\hat{g}^i(x)^\top \nabla h(x)\| \geq \varepsilon$ , for all  $i \in \bar{\mathbb{N}}$ ,  $\dot{\beta}$  becomes

$$\dot{\beta} \leq |\beta_d| \|\nabla h(x)\| \bar{f} - \kappa \beta_d^2 + \kappa \beta_d^2 \frac{\bar{h} \bar{g}}{\varepsilon}.$$

Therefore, since  $\bar{h} \bar{g} < \varepsilon$ , there exists a constant  $\epsilon$  such that  $\epsilon = 1 - \frac{\bar{h} \bar{g}}{\varepsilon} > 0$ , leading to

$$\dot{\beta} \leq -|\beta_d| (\epsilon \kappa |\beta_d| - \bar{h} \bar{f}). \quad (12)$$

Note that the latter inequality holds regardless of the index  $i$ . We claim now that (12) implies the boundedness of  $\beta$ . Assume that this is not the case, and that  $\lim_{t \rightarrow t_{\max}} \beta(h(x(t))) = \infty$ , which, in view of (7) and (9), implies that  $\lim_{t \rightarrow t_{\max}} |\beta_d(t)| = \infty$ . Hence, for every positive constant  $\gamma > 0$ , there exists a time instant  $t_\gamma \in (t_0, t_{\max})$  such that  $|\beta_d(t)| > \gamma$  for all  $t > t_\gamma$ . Consequently and since  $\bar{f}$  and  $\epsilon, \kappa$  are positive constants, we conclude from (12) that there exists a time instant  $t' \in [t_0, t_{\max})$  such that  $\beta(t) < 0$  for all  $t > t'$ , which leads to a contradiction. We conclude, therefore, that there exists a constant  $\bar{\beta}$  such that  $\beta(h(x(t))) \leq \bar{\beta}$ , for all  $t \in [t_0, t_{\max})$ , implying  $h(x(t)) \geq \underline{h} := \alpha_1^{-1} \left( \frac{1}{\bar{\beta}} \right)$ , for all  $t \in [t_0, t_{\max})$ , which dictates the boundedness of the system in a compact set  $x(t) \in \bar{\mathcal{C}} \subset \text{Int}(\mathcal{C})$ , for all  $t \in [t_0, t_{\max})$ . One can hence conclude from [27, Theorem 2.1.4] that  $t_{\max} = \infty$ , and that  $x(t) \in \bar{\mathcal{C}} \subset \text{Int}(\mathcal{C})$ , for all  $t \geq t_0$ . Moreover, since  $h(x(t)) \geq \underline{h}$ , for all  $t \in [t_0, t_{\max})$ , (9) and (11) imply the

boundedness of  $u(x(t), t)$ , for all  $t \geq t_0$ , leading to the conclusion of the proof.

Loosely speaking, the system will remain safe if the approximation error  $\tilde{g}$  is small enough, as quantified by the controllability constant  $\varepsilon$  in (10a). In Section V we provide an algorithm that deals with the case when (10a) might no longer hold. Moreover, it can be argued that Theorem 2 is overly conservative; the conditions (10) need only hold close to the boundary of  $\mathcal{C}$ , which gives rise to *local barrier control*, developed next.

### Local Barrier Control

The conditions  $\|\hat{g}^i(x)^\top \nabla h(x)\| \geq \varepsilon$ ,  $\bar{g} \bar{h} < \varepsilon$  imposed in Th. 2 are quite restrictive, since they have to hold for all  $x \in \text{Int}(\mathcal{C})$ , and  $i \in \bar{\mathbb{N}}$ . Moreover, note that (11) suggests that the controller is “enabled” regardless of the distance of  $x$  from  $\partial\mathcal{C}$ , which might yield a conservative system trajectory. This problem is alleviated in [6] by achieving condition (8), where  $\beta$  is allowed to grow when  $x$  is far away from  $\partial\mathcal{C}$ . As mentioned before, the system dynamics are unknown and hence such a condition cannot be achieved. Nevertheless, we show in this section that the control law can be designed to be “enabled” only close to the unsafe boundary  $\partial\mathcal{C}$ , where the proximity can be chosen by the user/designer. This allows the choice of any continuous nominal controller  $u_n(x)$  away from the boundary, which might be responsible for some (potentially unsafe) desired task. Moreover, we relax the strict conditions of Th. 2 to hold only close to the boundary.

Let the proximity from  $\partial\mathcal{C}$  be defined in terms of the positiveness of  $h(x)$ . That is, for a given constant  $\mu > 0$ , we want the controller to act only when  $0 < h(x) \leq \mu$ , which defines the set  $\mathcal{C}_\mu := \{x \in \mathbb{R}^n : h(x) \in (0, \mu]\}$ . To this end, we define the switching signal  $\sigma_\mu : [0, \infty) \rightarrow [0, 1]$ , with  $\sigma_\mu(h) = 0$  if  $h \geq \mu$ ,  $\sigma_\mu(h) = \phi_\mu$  if  $h \in [0, \mu]$ , and  $\sigma_\mu(h) = 1$  if  $h \leq 0$ , where  $\phi_\mu : [0, \infty) \rightarrow [0, 1]$  is any *decreasing* continuous function satisfying  $\phi_\mu(0) = 1$  and  $\phi_\mu(\mu) = 0$ , for some  $\mu > 0$ . The design parameters  $\phi_\mu$  and  $\mu$  tune how “aggressive” the system behaves to ensure safety. The control law is now designed as

$$u := u(x, t) = u_n(x) - \kappa \sigma_\mu(h(x)) \beta_d \frac{\hat{g}^i(x)^\top \nabla h(x)}{\|\hat{g}^i(x)^\top \nabla h(x)\|^2}, \quad (13)$$

where  $u_n(x)$  is a nominal continuous controller and  $\kappa$  is a positive constant control gain. Similarly to the proof of Th. 2, we establish the existence and uniqueness of an absolutely continuous solution  $x(t)$  evolving in  $\text{Int}(\mathcal{C})$ , for  $t \in [t_0, t_{\max})$ , for a positive time instant  $t_{\max} > t_0$ . Given  $\mu > 0$ , define the set  $K_\mu := \{i \in \bar{\mathbb{N}} : \exists[\tau_1, \tau_2] \subseteq [t_0, t_{\max})$ , with  $\tau_1 \in [t_i, t_{i+1})$  s.t.  $x(t) \in \mathcal{C}_\mu$ , for all  $t \in [\tau_1, \tau_2]\}$ , where  $t_i$  are the update instants from Section III. The set  $K_\mu$  contains the time index of the last update before entering the set  $\mathcal{C}_\mu$  as well as the time indices of the updates while in the set  $\mathcal{C}_\mu$  (see the purple points of the system trajectory in Fig. 1). Note that  $K_\mu$  is not empty, unless  $x(t) \in \text{Int}(\mathcal{C}) \setminus \mathcal{C}_\mu$  (i.e.,  $h(x(t)) \geq \mu$  for all  $t \geq t_0$ ). We are now ready to state the main results of this section.

**Theorem 3.** *Let a system evolve according to (2) and a set  $\mathcal{C}$  satisfying  $x(t_0) \in \text{Int}(\mathcal{C})$  for some  $t_0 \geq 0$ . Let functions  $h : \text{Int}(\mathcal{C}) \rightarrow [0, \infty)$ ,  $\beta : (0, \infty) \rightarrow \mathbb{R}$  satisfying (3) and (7), (9), respectively, and a constant  $\mu' \in (0, \mu)$ . Assume that*

$$\|\hat{g}^i(x)^\top \nabla h(x)\| \geq \varepsilon, \quad (14a)$$

$$\bar{g}_{\mu'} \bar{h}_{\mu'} < \varepsilon \sigma_{\mu\mu'} \quad (14b)$$

for a positive constant  $\varepsilon$ , for all  $i \in K_{\mu'}$  and  $x \in \mathcal{C}_{\mu'}$ , where  $\sigma_{\mu\mu'} := \sigma_{\mu}(\mu')$ ,  $\bar{h}_{\mu'} := \sup_{x \in \mathcal{C}_{\mu'}} \|\nabla h(x)\|$  and  $\bar{g}_{\mu'} := \sup_{\substack{x \in \mathcal{C}_{\mu'} \\ i \in K_{\mu'}}} \|\hat{g}^i(x)\|$ . Under Assumptions 1, 2, the control law

(13) guarantees that  $x(t) \in \text{Int}(\mathcal{C})$ , for all  $t > t_0$ , and the boundedness of all closed loop signals.

*Proof:* Since we have established the existence of a solution  $x(t) \in \text{Int}(\mathcal{C})$ , for a time interval  $t \in [t_0, t_{\max})$ , assume that  $\lim_{t \rightarrow t_{\max}} h(x(t)) = 0$ , i.e., the system converges to the boundary of  $\mathcal{C}$  as  $t \rightarrow t_{\max}$ , implying  $\lim_{t \rightarrow t_{\max}} \beta(h(x(t))) = \infty$ . Let any  $t' \in [t_0, t_{\max})$  such that  $x(t) \in \mathcal{C}_{\mu'}$  for all  $t \in [t', t_{\max})$ , and  $x(t) \in \bar{\mathcal{C}} := \{x \in \mathbb{R}^n : h(x) \geq \min_{t \in [t_0, t']} \{h(x(t))\} > 0\}$ , for all  $t \in [t_0, t']$ . Hence, it holds  $0 < h(x(t)) \leq \mu' < \mu$  and  $\sigma_{\mu}(h(x(t))) \geq \sigma_{\mu\mu'} := \sigma_{\mu}(\mu') > 0$ , for all  $t \in [t', t_{\max})$ . Moreover, note that  $t_i \in [t', t_{\max})$  implies  $i \in K_{\mu'}$ , which is non-empty since  $x(t) \in \mathcal{C}_{\mu'}$ , and consequently,  $\|\hat{g}^i(x(t))^\top \nabla h(x(t))\| \geq \varepsilon$ ,  $\bar{g}_{\mu'} \bar{h}_{\mu'} < \varepsilon \sigma_{\mu\mu'}$ , for all  $t \in [t', t_{\max})$ . By recalling that  $\sigma_{\mu}(h) \leq 1$ ,  $\hat{\beta}$  becomes

$$\hat{\beta} \leq \beta_d \nabla h(x)^\top f_n(x) - \kappa \sigma_{\mu\mu'} \beta_d^2 + \kappa \beta_d \frac{\|\hat{g}^i(x)^\top \nabla h(x)\|}{\|\hat{g}^i(x)^\top \nabla h(x)\|}$$

for all  $t \in [t', t_{\max})$ , where  $f_n(x) := f(x) + g(x)u_n(x)$ . Since  $f, g, u_n$  are continuous functions and  $x(t) \in \text{Int}(\mathcal{C})$  for all  $t \in [t_0, t_{\max})$ , there exists a constant  $\bar{f}_n$ , independent of  $t_{\max}$ , satisfying  $\|f_n(x(t))\| \leq \bar{f}_n$ , for all  $t \in [t_0, t_{\max})$ . By also using  $\|\hat{g}^i(x(t))^\top \nabla h(x(t))\| \geq \varepsilon$ , for all  $t \in [t', t_{\max})$ , we obtain

$$\hat{\beta} \leq |\beta_d| \|\nabla h(x)\| \bar{f}_n - \kappa \sigma_{\mu\mu'} \beta_d^2 + \kappa \beta_d \frac{\bar{h}_{\mu'} \bar{g}_{\mu'}}{\varepsilon},$$

for all  $t \in [t', t_{\max})$ . By setting  $\epsilon_{\mu'} := \sigma_{\mu\mu'} - \frac{\bar{h}_{\mu'} \bar{g}_{\mu'}}{\varepsilon} > 0$ , we obtain

$$\hat{\beta} \leq -|\beta_d| (\epsilon_{\mu'} |\beta_d| \kappa - \bar{h}_{\mu'} \bar{f}_n)$$

for all  $t \in [t', t_{\max})$ . By invoking similar arguments as in the proof of Th. 2, we conclude that  $x(t) \in \bar{\mathcal{C}} \subset \text{Int}(\mathcal{C})$ . By also using  $x(t) \in \bar{\mathcal{C}}$ , for all  $t \in [t_0, t']$  and the compactness of the latter, we conclude the boundedness of  $x(t)$  and  $\beta(h(x(t)))$ , for all  $t \in [t_0, t_{\max})$ . From [27, Th. 2.1.4], we conclude that  $t_{\max} = \infty$ , and the boundedness of  $x(t)$ ,  $\beta(h(x(t)))$ ,  $u(x(t), t)$  for all  $t \in [t_0, \infty)$ .

Intuitively, the condition  $\bar{g}_{\mu'} \bar{h}_{\mu'} < \varepsilon \sigma_{\mu\mu'}$  of Th. 3 is implicitly connected to the frequency of measurements  $\{x_i, \hat{x}_i, u_i\}$  the system obtains. More specifically, note first that  $\|\hat{g}^i(x(t_i))\| \leq \lim_{t \rightarrow t_i^-} \|\hat{g}^i(x(t_i))\|$ , for all  $i \in \bar{\mathbb{N}}$ , i.e., the estimation  $g^i(x)$  of  $g(x)$  improves with every update. Hence, the condition  $\bar{g}_{\mu'} \bar{h}_{\mu'} < \varepsilon \sigma_{\mu\mu'}$  simply implies that the system will have obtained sufficiently enough measurements

such that it obtains an accurate enough estimate  $\hat{g}^i(x)$  of  $g(x)$  from Alg. 1 before it reaches  $\partial(\mathcal{C})$ . An explicit relation is out of the scope of this paper and is left for future work.

## V. LOSS OF CONTROLLABILITY

In this section we provide an algorithm that considers cases where  $\|\hat{g}^i(x)^\top \nabla h(x)\|$  can become arbitrarily small, relaxing thus the respective assumption in Th. 2 and 3.

The framework we follow in order to tackle such cases is an online switching mechanism that computes locally alternative barrier functions  $h_\nu$ , defining new safe sets  $\mathcal{C}_\nu := \{x \in \mathbb{R}^n : h_\nu(x) \geq 0\} \subset \mathcal{C}_{\nu-1}$ , for  $\nu \geq 2$ , with  $h_1 := h$ ,  $\mathcal{C}_1 := \mathcal{C}$ . More specifically, at a point  $x_c$  where  $\|\hat{g}^i(x_c)^\top \nabla h(x_c)\|$  becomes too small, the algorithm looks for an alternative function  $h_2$  satisfying  $\mathcal{C}_2 \subset \mathcal{C}$  and for which  $\|\hat{g}^i(x_c)^\top \nabla h_2(x_c)\|$  is sufficiently large. The controller switches locally to  $h_2$  until  $\|\hat{g}^i(x)^\top \nabla h(x)\|$  becomes sufficiently large again. In case the system navigates along the line  $\|\hat{g}^i(x)^\top \nabla h(x)\| = 0$  to a region where  $\|\hat{g}^i(x)^\top \nabla h_2(x)\|$  is also small, the procedure is repeated and a new  $h_3$  is computed. Note that, since we desire  $\hat{g}^i(x)$  to be close to  $g(x)$ , and hence the interval  $G^i(x)$  to be small (see Lemma 3), choosing a different  $\hat{g}^i(x)$  from  $G^i(x)$  is not expected to significantly modify the term  $\|\hat{g}^i(x)^\top \nabla h(x)\|$ .

The aforementioned procedure is described more formally in the algorithm SafetyAdaptation (Alg. 2). More specifically, for a given  $j$ , each  $\rho_j$  indicates whether the system is close to the set where  $\|\hat{g}^i(x)^\top \nabla h_j(x)\| = 0$ . If that's the case ( $\rho_j = 0$ ), then a new function  $h_{j+1}$  is computed such that  $\mathcal{C}_{j+1} \subset \mathcal{C}_j$ , and in the switching point it holds that  $h_{j+1}(x) > 0$  and  $\|\hat{g}^i(x)^\top \nabla h_{j+1}(x)\|$  is sufficiently large;  $h_{j+1}(x)$  is used then in the control law. If  $\|\hat{g}^i(x)^\top \nabla h_\nu(x)\|$  becomes sufficiently large, for some  $\nu < j$ , then  $j$  is set back to  $\nu$ , and  $h_\nu(x)$  can be safely used in the control law again. We also impose a hysteresis mechanism for the switching of the constants  $\rho_\nu$  (lines 7, 13) through the parameters  $\bar{\varepsilon}, \underline{\varepsilon}$ .

The reasoning behind Alg. 2 is the following. By appropriately choosing the functions  $h_\nu(x)$ , the solutions of  $\|\hat{g}^i(x)^\top \nabla h_\nu(x)\| = 0$  form curves of measure zero. Hence, the intersection of  $n$  such lines will be a single point and hence the employment of a newly computed  $h_{n+1}$  will drive the system away from that point, resetting the algorithm. More formally, there is no time instant  $t \geq t_0$  such that  $\|\hat{g}^i(x(t))^\top \nabla h_\nu(x)\| \leq \underline{\varepsilon}$  for  $\nu \geq n+1$ , implying that the iterator variable  $j$  of Alg. 2 will never exceed  $n+1$ . A formal proof for the aforementioned reasoning is, however, not in the scope of this paper and consists part of future work. The formal definition of the control law is

$$u = u_n(x) - \kappa \sigma_{\mu}(h(x)) \beta_d u_b(x, t) \quad (15a)$$

$$u_b := \sum_{\nu=1}^{n+1} \rho_\nu \prod_{j=1}^{\nu-1} (1 - \rho_j) \frac{\hat{g}^i(x)^\top \nabla h_\nu(x)}{\|\hat{g}^i(x)^\top \nabla h_\nu(x)\|^2}, \quad (15b)$$

for all  $t \in [t_i, t_{i+1})$ ,  $i \in \bar{\mathbb{N}}$ , with  $h_1 = h$  and  $\sigma_{\mu}$  as in (13).

The SafetyAdaptation algorithm is run separately for each time interval  $[t_i, t_{i+1})$ ,  $i \in \bar{\mathbb{N}}$ . That is, when a new measurement  $(x(t_{i+1}), \hat{x}(t_{i+1}), u(t_{i+1}))$  is received, the estimation

of  $g(x)$  is updated, a new  $\hat{g}^{i+1}$  is computed by Alg. 1, and SafetyAdaptation is reset ( $j$  and  $\rho_\iota$  are reset as in lines 1–2).

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**Algorithm 2** SafetyAdaptation( $g^i, h, \bar{\varepsilon}, \underline{\varepsilon}$ )

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1:  $\rho_\iota \leftarrow 1, \forall \iota \in \{1, \dots, n+1\}$ ;
2:  $j \leftarrow 1; h_1 \leftarrow h$ ;
3: while True do
4:   if  $\|\hat{g}^i(x)^\top \nabla h_j(x)\| \leq \underline{\varepsilon} \wedge \rho_j = 1$  then
5:      $x_c \leftarrow x; \rho_j \leftarrow 0$ ;
6:     Find  $h_{j+1}$  such that
       1)  $\mathcal{C}_{j+1} \subset \mathcal{C}_j$ 
       2)  $h_{j+1}(x_c) > 0$ 
       3)  $\|\hat{g}^i(x_c)^\top \nabla h_{j+1}(x_c)\| \geq \gamma_{j+1}$ ;
7:      $j \leftarrow j + 1$ ;
8:   end if
9:   for  $\iota \in \{1, \dots, j-1\}$  do
10:    if  $\|\hat{g}^i(x)^\top \nabla h_\iota(x)\| > \bar{\varepsilon} \wedge \rho_\iota = 0$  then
11:       $\rho_\iota \leftarrow 1; j \leftarrow i$ ;
12:      Break;
13:    end if
14:  end for
15:  Apply (15)
16: end while

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Alg. 2 imposes an extra, state-dependent switching to the closed-loop system, which can be written as

$$\dot{x} = f(x) + g(x)u_n - \kappa\sigma_\mu(h(x))g(x)\beta_d u_\iota(x, t), \quad (16)$$

where  $u_\iota := \frac{\hat{g}^i(x)^\top \nabla h_\iota(x)}{\|\hat{g}^i(x)^\top \nabla h_\iota(x)\|^2}$ , for some  $i \in \bar{\mathbb{N}}, \iota \in \mathbb{N}$ . The switching regions are not pre-defined, but detected online based on the trajectory of the system (line 4 of Alg. 2). Moreover, by choosing  $\gamma_j > \underline{\varepsilon}$ , for all  $j \geq 2$ , we guarantee that the switching does not happen continuously, and hence the solution of (16) is well-defined in  $[t_0, t_{\max})$  for some  $t_{\max} > t_0$ , satisfying  $x(t) \in \text{Int}(\mathcal{C})$ , for all  $t \in [t_0, t_{\max})$ .

The results of this section are summarized as follows.

**Theorem 4.** *Let a system evolve according to (2) and a set  $\mathcal{C}$ , satisfying  $x(t_0) \in \text{Int}(\mathcal{C})$  for a positive  $t_0 \geq 0$ . Let functions  $h : \text{Int}(\mathcal{C}) \rightarrow [0, \infty)$ ,  $\beta : (0, \infty) \rightarrow \mathbb{R}$  satisfying (3) and (7), (9), respectively, and a positive constant  $\mu' \in (0, \mu)$ . Assume that  $\bar{g}_{\mu'} \bar{h}_{\mu'} < \varepsilon \sigma_{\mu\mu'} := \sigma_\mu(\mu')$ , for a positive constant  $\varepsilon$ , for all  $i \in K_{\mu'}$  and  $x \in \mathcal{C}_{\mu'}$ , where  $\bar{h}_{\mu'} := \sup_{x \in \mathcal{C}_{\mu'}} \|\nabla h(x)\|$  and  $\bar{g}_{\mu'} := \sup_{\substack{x \in \mathcal{C}_{\mu'} \\ i \in K_{\mu'}}} \|\hat{g}^i(x)\|$ . Let Assumptions 1, 2 hold and consider the control law (15), with  $\rho_\iota$  computed as in Alg. 2. Further assume that there is no time instant  $t \geq t_0$  such that  $\|\hat{g}^i(x(t))^\top \nabla h_\iota(x(t))\| \leq \underline{\varepsilon}$  for  $\iota \geq n+1$ . Then  $x(t) \in \text{Int}(\mathcal{C})$  and all closed loop signals remain bounded, for all  $t > t_0$ .*

*Proof:* The proof follows from the fact that at the switching regions, i.e., when  $\|\hat{g}^i(x)^\top \nabla h_j(x)\| \leq \underline{\varepsilon}$ , it holds that  $h_j(x_c) > h_{j+1}(x_c) > 0$  for  $j \in \{1, \dots, n\}$ , and by applying similar arguments as in the proof of Th. 3 for each fixed  $j \in \{1, \dots, n+1\}$ .

## VI. SIMULATION RESULTS

We validate the proposed algorithm with a simulation example. More specifically, we consider an underactuated unmanned aerial vehicle (UAV) with state variables  $x = [x_1, \dots, x_6] = [p_x, p_y, \phi, v_x, v_y, \omega]^\top$  evolving subject to the dynamics  $\dot{p}_x = v_x, \dot{p}_y = v_y, \dot{\phi} = \omega$ , and

$$\begin{aligned} m\dot{v}_x &= -C_D^v v_x - u_1 \sin(\phi) - u_2 \sin(\phi) \\ m\dot{v}_y &= -(mg + C_D^v v_y) + u_1 \cos(\phi) + u_2 \cos(\phi) \\ 2I\dot{\omega} &= -C_D^\phi \omega - l u_1 + l u_2, \end{aligned}$$

where  $m = 1.25$ ,  $I = 0.03$  are the quadrotor's mass and moment of inertia, respectively,  $g = 9.81$  is the gravity constant,  $l = 0.5$  is the arm length, and  $C_D^v = 0.25$ ,  $C_D^\phi = 0.02255$  are aerodynamic constants. We consider that the UAV aims to track the helicoidal trajectory  $p_{x_r} := \frac{1}{2} \sin(\frac{3}{2}t)$ ,  $p_{y_r} := \frac{1}{2} \sin(\frac{3}{4}t)$  (see Fig. 2) via an appropriately designed nominal control input  $u_n$ . We wish to bound the vertical and angular velocities of the UAV through via the ellipsoid  $h(x) = 1 - [v_y, \omega + 0.5] \text{diag}\{7, 0.5\} [v_y, \omega + 0.5]^\top$ , which is pictured in Fig. 3. We use the local safety controller (15), with  $\beta = \frac{1}{h}$ ,  $\mu = 0.5$ , and  $\kappa = 1$ , while setting  $\underline{\varepsilon} = 0.2$ ,  $\bar{\varepsilon} = 0.6$  in Alg. 2. The data measurement and hence the execution of Alg. 1 occurred every 0.5 seconds. The simulation results from the initial condition  $[0, 0, 0, -0.1, 0, 0]^\top$  are illustrated in Figs. 2-4 for  $t = 10$  seconds. In particular, Fig. 2 depicts the reference helicoidal trajectory (red) and the system trajectory under the nominal  $u_n$  (blue) and local safety controller (15) (green); Fig. 3 depicts the barrier and local barrier function boundaries  $h(x) = 0$  (red),  $h(x) = \mu = 0.5$  (magenta), respectively, as well as the vertical and angular velocities under the nominal  $u_n$  (blue) and local safety controller (15) (green). One can verify that the system is successfully confined in the set  $\text{Int}(\mathcal{C})$  defined by  $h(x) > 0$ , verifying thus the theoretical findings. Finally, Fig. 4 depicts the resulting control input  $u(t)$  (top) and the approximation error 2-norm  $\|\tilde{g}^i(x)\|$  (bottom). From Fig. 4 it can be verified that the condition (14b) does not always hold and hence it is not necessary for Th. 3, 4. We note also that in this example  $\|\hat{g}^i(x)^\top \nabla h(x)\| \geq \underline{\varepsilon}$  is always satisfied.

## VII. CONCLUSION AND FUTURE WORK

We consider the safety problem for a class of nonlinear unknown systems. We propose a two-layered control solution, integrating approximation of dynamics from limited data with closed form nonlinear control laws using reciprocal barriers. Future efforts will be devoted towards extending the proposed framework to stabilization/tracking and provide conditions on the frequency of the measured data.

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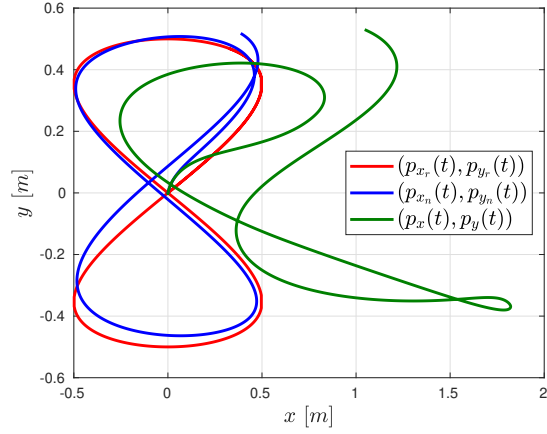


Fig. 2. The desired reference trajectory  $(p_{x_r}(t), p_{y_r}(t))$  (red), and the system trajectories  $(p_{x_n}(t), p_{y_n}(t))$  (blue),  $(p_x(t), p_y(t))$  (green) under  $u_n$  and (15), respectively, for  $t = 10$  seconds.

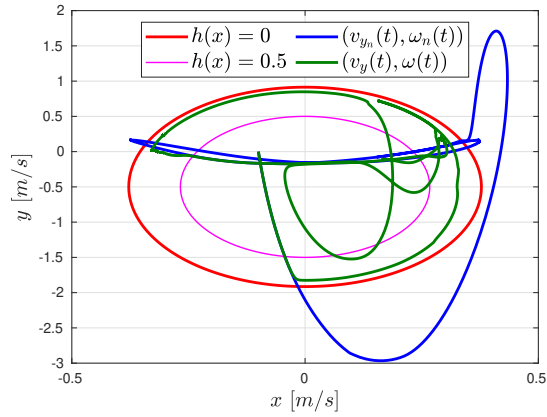


Fig. 3. The barrier levels  $h(x) = 0$  (red),  $h(x) = 0.5$  (magenta), and the system velocities  $(v_{y_n}(t), \omega_n(t))$  (blue),  $(v_y(t), \omega(t))$  (green) under  $u_n$  and (15), respectively, for  $t = 10$  seconds.

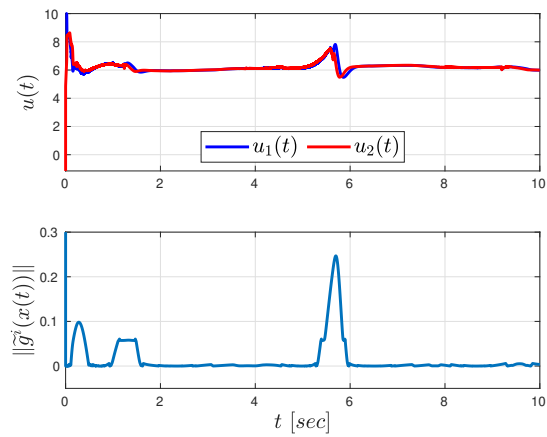


Fig. 4. Top: The resulting control signal  $u(t)$ ; Bottom: The error norm  $\|\tilde{g}^i(x(t))\|$  for  $t = 10$  seconds.

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